Monge-Ampère equation in hypercomplex geometry

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A thesis presented for the degree of Doctor of Philosophy
in the subject of Mathematics

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2021

Abstract

We provide number of results concerning the quaternionic Monge-Ampère equation. This is the type of partial differential equation which appears in the presence of the hyper-hermitian structure. For domains in the affine space we solve the Dirichlet problem for a continuous boundary data and the right hand side in L^p for $p > 2$. This generalizes many previous results on that problem. We show that this assumption on p is optimal. On compact hyperKähler with torsion manifolds we present the proof of the uniform a priori estimate for this equation.

Acknowledgments

I use this opportunity to express my deep gratitude to Sławomir Kołodziej for the guidance during the period of my studies. The time he has spent on discussions with me as well as his countless advices made a grave impact on me as a mathematician.

I am very grateful to Sławomir Dinew for all the support I have received from him. He always serves as a great source of motivation for me.

I would like to mention many encounters and the resulting idea exchanges with my Italian friend Daniele Angella, *grazie mille!*

During the studies and the preparation of this thesis I was supported by the Kartezjusz fellowship, granted under the program POWR.03.02.00-00-I001/16-00, and the National Science Center of Poland grant numbers 2017/27/B/ST1/01145, 2020/36/T/ST1/00334.

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Introduction

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1.1 Contents description

The subject of the thesis is the existence and regularity of solutions to the quaternionic Monge-Ampère equation, in both the local and the global setting. It is a counterpart of the real and complex PDEs playing the prominent roles in both analysis and geometry. It is an equation with a new form of non-linearity appearing naturally in the presence of certain geometric structure – the hyperhermitian one. All the presented results are contained in the series of three papers [Sr18, KS20, Sr19], one of which is co-authored with my supervisor. The research on this topic in the local setting was initiated by Alesker [A03a] and independently by Harvey and Lawson [HL09c]. Afterwards it was realized that the quaternionic Monge-Ampère operator plays a significant role in the geometry of hyperKähler with torsion manifolds, the type of geometry appearing on target spaces of certain $\sigma-$models in quantum mechanics, cf. [GP00]. This observation made Alesker and Verbitsky pose the analogue of the Calabi conjecture for those space, cf. [AV10]. This in turn can be reduced to solving the quaternionic Monge-Ampère equation on hyperKähler with torsion manifolds.

We briefly summarize the results, which are contained in [Sr18, KS20], for the flat case of $\mathbb{H}^n$, where $\mathbb{H}$ denotes the field of quaternions. Let $MA_{\mathbb{H}}$ denote the quaternionic Monge-Ampère operator in that space. Our main result is Theorem 5.3.1 below. It gives the existence and uniqueness of the solutions, in the weak distributional sense, to the Dirichlet problem

$$\begin{cases} MA_{\mathbb{H}}(u) = f \\ u_{|\partial D} = \phi \\ u \in \mathcal{QPSH}(D) \cap C^0(\overline{D}) \end{cases} .$$

In the above D is sufficiently convex and smooth, ϕ is continuous and $f \in L^p(D)$ for $p > 2$.

This is the most general result concerning the existence of the continuous solutions that is known. It generalizes the previous results due to Zhu [Z17] for the smooth data, Alesker [A03b] and Harvey and Lawson [HL09c, HL20] for the continuous ones and Wan [W20] for the densities f from $L^p(D)$ for $p \geq 4$. We show also that the bound for the exponent p for which there exist the continuous solutions is optimal. This in turn shows that Theorem 5.3.1 constitutes the analog of Alexandrov's result for the real Monge-Ampère equation, where $p \geq 1$, and Kołodziej's theorem for the complex Monge-Ampère equations, where $p > 1$.

The proof of Theorem 5.3.1 is based on the local C^0 estimate for the quaternionic Monge-Ampère equation. This in turn relies on the crucial result – the inequality between the capacity, naturally associated with the quaternionic Monge-Ampère operator, and the Lebesgue measure. The proof here is completely different from its complex analogue. First we compare the complex and quaternionic Monge-Ampère operators. This idea for the pair of complex and real Monge-Ampère operators is due to Cheng and Yau. Afterward we use a trick from [DK14] by noting that the class of functions on which the equation is elliptic contains in particular the plurisubharmonic functions. Finally we apply the Kołodziej's C^0 estimate for the complex Monge-Ampère equation. To make this as simple as possible, the proof of the C^0 estimate we give is not the repetition of the Kołodziej's proof in the complex case yet it crucially uses this result. It is not known whether one can obtain the C^0 estimate for the quaternionic Monge-Ampère equation without using this result.

The regularity of the solutions from Theorem 5.3.1 is given in Theorem 6.5, this is the contents of [KS20]. We show that if the boundary condition is of the class $C^{1,1}$ then the solution has to be of the class $C^{0,\alpha}$ with the bound on α in terms of p and n. The assumptions in this result are quite weak though as we require the density to be bounded near the boundary of the domain.

On compact HKT manifolds the quaternionic Monge-Ampère equation appears naturally. It encodes the possibility of obtaining any section of the canonical bundle of a hypercomplex manifold from a hyperKähler with torsion metric, cf. [AV10]. Those metrics generalize the well know hyperKähler metrics. The troubles with solving the equation in the global case rely on obtaining the a priori estimates at least up to the order $C^{2,\alpha}$. Partial progress was obtained in [AV10, AS13, A13]. So far the only estimate known in the general case, what we discuss in detail in Part III, is the uniform bound. It was obtained by Alesker and Shelukhin in [AS17]. We present, in our opinion, simpler proof of this result which appeared in [Sr19]. What is more important we improve the dependence of the estimate on the initial data, it now depends only on the L^p norm of the right hand side for a suitable p and the geometric quantities. This result is an analog of Yau's estimate on Kähler manifolds and Tosatti and Weinkove's estimate on hermitian manifolds for the complex Monge-Ampère equation.

The idea of the proof is as follows. We apply the Sobolev inequality for the solution. This requires the L^2 bound on the gradient of the solution, obtaining which is the main challenge. In order to accomplish a weaker estimate, dependent on the solution, we differentiate the equation and extract this weaker bound. This is done in a rather lengthy process of obtaining an iterative bound on quantities appearing due to the mentioned differentiation, see Chapter 8. As it turns out this allows us to obtain the desired C^0 bound after applying the Moser iteration method and some key ideas from [TW10a, TW10b]. The whole proof is strongly motivated by the latter paper.

The described results enrich the research on the fairly general class of PDEs and the associated pluripotential theory initiated by the papers [HL09a, HL09b, HL09c]. In the global case, the last two decades witnessed tremendous advances in geometric analysis on hermitian manifolds. The particular example are the Calabi-Yau type theorems for different classes of hermitian metrics [TW10b, GL10, SzTW17, TW17]. As we discuss in Chapter 7 the quaternionic Monge-Ampère equation on hyperKähler with torsion manifolds fits in that setting.

1.2 Dissertation organization

The organization of the text is as follows. In the next three chapters all the preliminaries are presented. Those are grouped into, receptively, completely elementary results from quaternionic linear algebra, geometric analysis on hermitian manifolds and pluripotential theory associated to the quaternionic Monge-Ampère operator. No claim on the originality is made there except for the contents of Section 4.6. The results described above for the Dirichlet problem are presented in Part II while those for the compact HKT manifolds in Part III. We tried to organize the thesis so that, in case of need, one can read those two parts separately. For reading Part II one needs Chapter 2, Chapter 4 as well as Section 3.3.2 from Chapter 3. For reading Part III Chapter 2 and Chapter 3 are required.

Chapter 2

Quaternionic linear algebra

This short chapter is meant to be a completely elementary introduction to the quaternionic linear algebra serving mostly the purpose of fixing the notation. All of the presented material is well known except, maybe, the Moore determinant which will be crucial for us later when defining the quaternionic Monge-Ampère operator. We do not always know the references containing proofs of the results as we present them, e.g. the relation between Pfaffian and Moore determinant, that is why we do not provide them for each result but we claim no originality. References for the quaternionic linear algebra that one can consult, in case of need, are [Su79, Zh97], especially the latter one. For hyperhermitian linear algebra the introductory section of [A03a] is probably the most adequate. We learned about the relation between Pfaffian and Moore determinant from [Dy70, Dy72]. For the discussion of determinants of quaternionic matrices we refer to [As96].

2.1 Quaternions

In this work we will face three number fields (and the corresponding geometries). These of real numbers - $\mathbb{R}$, complex ones - $\mathbb{C}$ and quaternions (formally not being a field of numbers) - $\mathbb{H}$. The last one is a skew-field, i.e. a ring satisfying all the axioms of the field except commutativity of multiplication, which will be denoted by

$$\mathbb{H} = \{x_0 + x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where $\mathfrak{i}^2 = \mathfrak{j}^2 = \mathfrak{k}^2 = -1$ and $\mathfrak{i}\mathfrak{j}\mathfrak{k} = -1$. The addition and multiplication being defined in the obvious way. For a quaternion

$$q = t + x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k}$$

we define its conjugate

$$\bar{q} = t - x\mathfrak{i} - y\mathfrak{j} - z\mathfrak{k}.$$

One can easily check that

$$q\bar{q} = \bar{q}q = t^2 + x^2 + y^2 + z^2$$

and

$$\overline{qp} = \bar{p}\bar{q}$$

for any $q, p \in \mathbb{H}$. This conjugation agrees with the complex conjugation for $\mathbb{C}$ embedded into $\mathbb{H}$ as described below.

The following embeddings of rings will be used

$$\mathbb{R} \ni x \longmapsto x + 0\mathfrak{i} \in \mathbb{C},$$
$$\mathbb{C} \ni x + y\mathfrak{i} \longmapsto x + y\mathfrak{i} + 0\mathfrak{j} + 0\mathfrak{k} \in \mathbb{H}.$$

With this identification one should remember that $z\mathfrak{j} = \mathfrak{j}\bar{z}$ for any complex z embedded in the mentioned way into $\mathbb{H}$.

We will deal with $\mathbb{H}^n$ considered as a **right** $\mathbb{H}$ vector space, formally a module over $\mathbb{H}$ but we will never use this formality. In particular $\mathbb{H}$ acts on itself from the right. In case of a vector space over a number field $\mathbb{K}(= \mathbb{R}, \mathbb{C})$ one also has to specify the side for the scalar action, this is usually the left action but in case of a field it does not matter due to the commutativity of multiplication. Taking that into an account we will treat real and complex vector spaces simultaneously as two sided. In case of quaternions we prefer to act from the right, and this is at least as common approach as the left one, in order to be able to identify a quaternionic matrix with an $\mathbb{H}$ linear mapping given by a multiplication of a vector in $\mathbb{H}^n$ from the **left** by this matrix.

As mentioned above, for a square quaternionic matrix

$$M = (m_{ij})_{i,j=1,\dots,n}$$

we have the associated $\mathbb{H}$ linear map

$$\mathbb{H}^n \ni q = (q_0, ..., q_{n-1}) \longmapsto Mq = (m_{1j}q_j, ..., m_{nj}q_j) \in \mathbb{H}^n \ .$$

Remark 2.1.1. *In the whole text we use Einstein summation convention.*

This is indeed $\mathbb{H}$ linear since

$$M(qp) = (M(q))\, p$$

for any $q \in \mathbb{H}^n$ and $p \in \mathbb{H}$. We would have a problem if we had chosen $\mathbb{H}^n$ to be the left quaternionic vector space since then there would be no reason to have

$$M(pq) = pM(q)$$

i.e. p would generally not commute with the entries of M. In our situation it may not, yet we still obtain $\mathbb{H}$ linearity. This correspondence between matrices and $\mathbb{H}$ linear mappings of $\mathbb{H}^n$ shows that left and right matrix inverses are the same. Consequently it makes sense to think about the group $Gl_n(\mathbb{H})$ of invertible quaternionic matrices.

Let $\mathcal{M}(n, \mathbb{H})$ be the set of all square quaternionic matrices of size n and suppose

$$q = (q_0, ..., q_{n-1}) \in \mathbb{H}^n$$

is an eigenvector for an eigenvalue $\lambda \in \mathbb{H}$ of $M \in \mathcal{M}(n, \mathbb{H})$, i.e.

$$M(q) = q\lambda.$$

From $\mathbb{H}$ linearity of M it follows that for any $p \in \mathbb{H}^* \, (= \mathbb{H} \setminus \{0\})$ one has

$$M(qp) = M(q)p = q\lambda p = qpp^{-1}\lambda p.$$

This shows also that any quaternion conjugate to λ, i.e. of the form

$$p\lambda p^{-1} \text{ for } p \in \mathbb{H}^*,$$

is an eigenvalue with an eigenvector quaternion proportional to q. Thus there is a sense to talk only about the conjugacy classes of eigenvalues for the quaternionic matrix, since otherwise there would be infinitely many of them in a generic case, as explained below.

If we denote by

$$conj(p) = \{qpq^{-1} \mid q \in \mathbb{H}^*\}$$

the conjugacy class of a quaternion p then $|conj(p)| = 1$ if and only if $p \in \mathbb{R}$ as an easy exercise shows, cf. [Zh97]. In other case $conj(p)$ is infinite but contains exactly two mutually conjugate purely complex elements, i.e. there is $\lambda \in \mathbb{C}$ such that $\lambda, \bar{\lambda} \in conj(p)$ and no other complex number belongs to $conj(p)$, cf. [Zh97]. Here we mean the usual embedding of $\mathbb{C}$ into $\mathbb{H}$ as above, we will not repeat this anymore. This also shows that there are at most n conjugacy classes of eigenvalues for $M \in \mathcal{M}(n, \mathbb{H})$. This is so because the mutually conjugate complex representatives are then eigenvalues for a complex square matrix of size $2n$ corresponding to a $\mathbb{C}$ linear mapping of $\mathbb{H}^n$, treated as the complex vector space, induced by M. We will not use this result in this generality but it will be discussed in more details in the one of the following sections.

2.2 Hyperhermitian linear algebra

Let us restrict to the class of the so called **hyperhermitian** matrices, i.e. those belonging to

$$Herm(\mathbb{H}, n) = \{M \in \mathcal{M}(n, \mathbb{H}) \mid : \ M = M^*\}$$

where

$$M^* = (\bar{M})^T = \overline{M^T}.$$

One should be careful as not all the operations known from the complex case commute, for example transposition and inverting. The following definition is similar to the one in the complex case.

Definition 2.2.1. *A map $h : \mathbb{H}^n \times \mathbb{H}^n \longrightarrow \mathbb{H}$ is called a hyperhermitian form if*

- $h(q + q', p) = h(q, p) + h(q', p)$ *for* $q, q', p \in \mathbb{H}^n$,
- $h(p, q\lambda) = h(p, q)\lambda$ *for* $p, q \in \mathbb{H}^n$ *and* $\lambda \in \mathbb{H}$,
- $h(q, p) = \overline{h(p, q)}$ *for* $p, q \in \mathbb{H}^n$.

We say that h is positive (non-negative) if $h(q, q) > 0$ ($h(q, q) \geq 0$) for any $q \in \mathbb{H}^n \setminus \{0\}$.

As we see in the proposition below, hyperhermitian matrices are precisely the matrices defining hyperhermitian forms in canonical basis of $\mathbb{H}^n$. There are far going analogies with the complex case, some of them are listed in the remark below.

Proposition 2.2.2. *The following map*

$$Inn : Herm(\mathbb{H}, n) \longrightarrow \{h \mid h \text{ is a hyperhermitian form on } \mathbb{H}^n\}$$

given by

$$Inn(M) = \{h(q, p) = \bar{q}^T Mp = \bar{q}_i m_{ij} p_j\}$$

is a bijection.

Remark 2.2.3.

- *For any* $C \in \mathcal{M}(n, \mathbb{H})$ *and* $M \in Herm(\mathbb{H}, n)$ *the matrix* C^*MC *is hyperhermitian. If in addition* $C \in Gl_n(\mathbb{H})$ *then the matrix* C^*MC *is the matrix of a hyperhermitian form* h*, defined by* M*, in a new basis. In this situation* C *is the transition matrix from the canonical basis to that new basis.*

- *One can always find an orthonormal basis of* $\mathbb{H}^n$ *for* $M \in Herm(\mathbb{H}, n)$ *in which* M *is real diagonal, i.e. for any* $M \in Herm(\mathbb{H}, n)$ *there is*

$$P \in Sp(n) = \{A \in Gl_n(\mathbb{H}) \mid : \ AA^* = id_n\} = \{A \in Gl_n(\mathbb{H}) \mid : \ A^*(\bar{q}_i p_i)A = \bar{q}_i p_i\}$$

  such that $P^*MP = D$ *where* D *is real diagonal.*

  The last description of $Sp(n)$ *tells us that this is a group of matrices preserving the standard quaternionic hyperhermitian product* $\overline{q_i}p_i$*.*

- *It follows that the (classes of) eigenvalues of* M *are all real. This will be discussed in more details in the next section.*

2.3 Determinants and canonical embeddings of matrices

The purpose of this section is to remind some embeddings of the space of complex matrices into real ones and to extend this notion to the pair of quaternionic and complex matrices. This is related to the problem of defining a determinant operator for elements of $\mathcal{M}(n, \mathbb{H})$. We follow the notation of [As96] closely. Let $\mathcal{M}(n, \mathbb{R})$ and $\mathcal{M}(n, \mathbb{C})$ be the sets of real and complex square matrices of size n.

One can identify $\mathbb{C}$ and $\mathbb{R}^2$, via an isomorphism of real vector spaces, in a standard way

$$\phi_1 : \mathbb{C} \ni x + y\mathfrak{i} \longmapsto (x, y) \in \mathbb{R}^2.$$

This isomorphism of real vector spaces applied coordinatewise gives rise to the following isomorphism

$$\phi_n : \mathbb{C}^n \ni (x_0 + y_0\mathfrak{i}, ..., x_{n-1} + y_{n-1}\mathfrak{i}) \longmapsto (x_0, ..., x_{n-1}, y_0, ..., y_{n-1}) \in \mathbb{R}^{2n}. \tag{I.2.0}$$

Any

$$M = A + \mathfrak{i}B \in \mathcal{M}(n, \mathbb{C})$$

defines a $\mathbb{C}$ linear, so in particular $\mathbb{R}$ linear, map of $\mathbb{C}^n$ consequently also an $\mathbb{R}$ linear map of $\mathbb{R}^{2n}$ - via an isomorphism ϕ_n. This induced map is given explicitly by

$$\phi_n \circ M \circ \phi_n^{-1}.$$

Lemma 2.3.1. *The matrix, say* $\Phi_n(M)$*, of this* $\mathbb{R}$ *linear map of* $\mathbb{R}^{2n}$*, in the standard basis is given explicitly by*

$$\Phi_n(A + \mathfrak{i}B) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{I.2.1}$$

Proof. Take $e_j \in \mathbb{R}^{2n}$ the j'th vector from the canonical basis. Note that

$$\phi_n^{-1}(e_j) = \tilde{e}_j \text{ for } 0 \le j \le n-1,$$
$$\phi_n^{-1}(e_j) = \mathfrak{i}\tilde{e}_{j-n} \text{ for } n \le j \le 2n-1,$$

where $\tilde{e_j}$ is the canonical basis of $\mathbb{C}^n$.

For $0 \le j \le n-1$ we see that $\phi_n^{-1}(e_j)$ is mapped by M to the j'th column of M which is j'th column of A plus $\mathfrak{i}$ times the j'th column of B. Taking ϕ_n of this sum gives the j'th column of $\Phi_n(M)$. For other j's the reasoning is similar. □

In this way we have obtained a monomorphism of real algebras

$$\Phi_n : \mathcal{M}(n, \mathbb{C}) \longrightarrow \mathcal{M}(2n, \mathbb{R}).$$

Lemma 2.3.2. *The image of* Φ_n *is*

$$\Phi_n(\mathcal{M}(n, \mathbb{C})) = \{N \in \mathcal{M}(2n, \mathbb{R}) \mid NI_n = I_nN\}$$

where $I_n = \Phi_n(\mathfrak{i}id_n)$ *and* id_n *is the identity matrix of size* n.

Remark 2.3.3. *The notation from this lemma is fixed for the whole text, i.e.* I_n *will never denote an identity matrix and* id_n *always denotes an identity matrix of size* n.

Proof. Note that $\mathbb{C}^n$ is isomorphic, via ϕ_n, with $\mathbb{R}^{2n}$ treated as a complex vector space, with multiplication by $\mathfrak{i}$ given by the map induced by the matrix I. Then $N \in \mathcal{M}(2n, \mathbb{R})$ corresponds to a $\mathbb{C}$ linear mapping of this complex vector space if and only if it commutes with multiplication by $\mathfrak{i}$ in $\mathbb{R}^{2n}$. □

Proposition 2.3.4. *For any* $M \in \mathcal{M}(n, \mathbb{C})$

$$|\det M|^2 = \det \Phi_n(M).$$

Proof. By adding suitable raws and columns one can see that

$$\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det \begin{pmatrix} A + \mathfrak{i}B & \mathfrak{i}(A + \mathfrak{i}B) \\ B & A \end{pmatrix} = \det \begin{pmatrix} A + \mathfrak{i}B & 0 \\ B & A - \mathfrak{i}B \end{pmatrix}$$

$$= \det \begin{pmatrix} M & 0 \\ B & \overline{M} \end{pmatrix} = \det M \cdot \det \overline{M} = \det M \cdot \overline{\det M} = |\det M|^2.$$

□

This result has a more precise form for a special type of matrices. This reasoning will give us an essential insight into how to define the determinant for some quaternionic matrices – as we will see in the next paragraph. Suppose

$$M = A + \mathfrak{i}B \in Herm(\mathbb{C}, n),$$

i.e.

$$\overline{M}^T = M.$$

This means that

$$A^T = A, B^T = -B.$$

Consequently $A \in Herm(\mathbb{R}, n)$ and B is skew-symmetric. This allows us to note the following.

Lemma 2.3.5. *For $M = A + \mathfrak{i}B \in Herm(\mathbb{C}, n)$ the matrix*

$$\Phi_n(M)$$

is symmetric while

$$I_n\Phi_n(M)$$

is skew-symmetric.

Proof. We show only the second observation, note that

$$I_n\Phi_n(M) = \Phi_n(\mathfrak{i}M) = \Phi_n(-B + \mathfrak{i}A) = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}.$$

Now from the properties of A and B it follows that

$$(I_n\Phi_n(M))^T = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}^T = \begin{pmatrix} B & A \\ -A & B \end{pmatrix} = -I_n\Phi_n(M).$$

□

Another important fact is the following result about the determinant of a skew-symmetric matrices. As far as we know it is due to Cayley, cf. [C1847]. We state it in modern language.

Theorem 2.3.6 (Cayley)**.** *There exists a polynomial Pf, with real coefficients, of degree n on the space of skew-symmetric complex matrices of size $2n$ such that*

$$det = Pf^2$$

as polynomials on this space.

The polynomial Pf from the theorem above, called Pfaffian, is defined up to the sign. From the reason which will be clear when defining the quaternionic Monge-Ampère operator in terms of product of currents we make the following choice.

Definition 2.3.7. *For a skew-symmetric matrix*

$$M = (m_{ij})_{i,j=1,\dots,2n} \in \mathcal{M}(2n, \mathbb{C})$$

we define the Pfaffian of M as

$$Pf(M)e_1 \wedge \dots \wedge e_{2n} = \frac{1}{n!}\left(\sum_{i<j} m_{ij}e_i \wedge e_j\right)^n$$

where e_i is the canonical basis of $\mathbb{C}^{2n}$ and the power in the exponent means exterior power - which is the convention in the whole text for the exterior power of exterior forms.

As det and Pf^2 are homogeneous polynomials of degree n the, non instructive, proof of Theorem 2.3.6 reduces to checking the equality on sufficiently many matrices of one's choosing. Having all of this settled the announced improvement of Proposition 2.3.4 is as follows.

Proposition 2.3.8. *For any $M \in Herm(\mathbb{C}, n)$*

$$\det M = Pf(I_n) \cdot Pf\left(I_n \Phi_n(M)\right)$$

in particular

$$(\det M)^2 = \left(Pf\left(I_n \Phi_n(M)\right)\right)^2.$$

Remark 2.3.9. *The proof of the first formula is by showing equality of homogeneous polynomials of degree n on $Herm(\mathbb{C}, n)$, like in the case of Theorem 2.3.6. The second formula follows. We need the normalization by $Pf(I_n)$ due to the fact that with the Definition 2.3.7 of Pfaffian as above one should convert $n \times n$ hermitian matrices into $n \times n$ matrices of 2×2 blocks rather than 2×2 matrices of $n \times n$ blocks but the latter is more convenient for latex typesetting.*

Let us now turn to the quaternionic case. We try to repeat the story for $\mathbb{H}$ with the isomorphism of (right) complex vector spaces

$$\psi_1 : \mathbb{H} \ni z_0 + \mathfrak{j} z_1 \longmapsto (z_0, z_1) \in \mathbb{C}^2$$

where, for

$$q_0 := q = x_0 + x_1 \mathfrak{i} + x_2 \mathfrak{j} + x_3 \mathfrak{k},$$

we put

$$q_0 = x_0 + x_1 \mathfrak{i} + \mathfrak{j}(x_2 - x_3 \mathfrak{i}) := z_0 + \mathfrak{j} z_1.$$

It extends to

$$\psi_n : \mathbb{H}^n \ni (z_0 + \mathfrak{j} z_1, ..., z_{2n-2} + \mathfrak{j} z_{2n-1}) \longmapsto (z_0, ..., z_{2n-2}, z_1, ..., z_{2n-1}) \in \mathbb{C}^{2n}.$$

This is one of the two natural identifications, suited for considering right quaternionic vector spaces. This time to an $\mathbb{H}$ linear mapping of $\mathbb{H}^n$ represented by

$$M = A + \mathfrak{j} B,$$

for $A, B \in \mathcal{M}(n, \mathbb{C})$, we associate a mapping of $\mathbb{C}^{2n}$. It is defined formally as

$$\psi_1 \circ M \circ \psi_1^{-1},$$

with the matrix in standard basis

$$\Psi_n(A + \mathfrak{j} B).$$

One obtains the following.

Lemma 2.3.10. *For any $M = A + \mathfrak{j} B$ one has*

$$\Psi_n(A + \mathfrak{j} B) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}. \tag{I.2.2}$$

The map

$$\psi : \mathcal{M}(n, \mathbb{H}) \longrightarrow \mathcal{M}(2n, \mathbb{C})$$

is a monomorphism of complex algebras.

Proof. Reasoning is just an adjustment of the previous one in real-complex case. □

Note that $\mathfrak{j}$ acting on $\mathbb{H}^n$, as we agreed from the right, defines a mapping of $\mathbb{C}^{2n}$ given by

$$\phi_n \circ \mathfrak{j} \circ \phi_n^{-1} : \mathbb{C}^{2n} \ni (z_0, ..., z_{2n-2}, z_1, ..., z_{2n-1}) \longmapsto (-\bar{z}_1, ..., -\bar{z}_{2n-1}, \bar{z}_0, ..., \bar{z}_{2n-2}) \in \mathbb{C}^{2n}.$$

As it is easy to see, this map equals to $\bar{\ } \circ I_n$ and is complex anti-linear.

With multiplication by $\mathfrak{j}$ defined by the map $\bar{\ } \circ I_n$ and by $\mathfrak{k}$ given by $\mathfrak{i} \circ \mathfrak{j}$ we obtain that the complex isomorphism ψ_n is an $\mathbb{H}$ linear isomorphism between $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ for $\mathbb{C}^{2n}$ treated as a right $\mathbb{H}$ vector space with an action of $\mathbb{H}$ as just described. We point out that in the above definition of multiplication by $\mathfrak{k}$ give a vector in $\mathbb{C}^{2n}$ one fist multiplies by $\mathfrak{i}$ and then applies the endomorphism $\bar{\ } \circ I_n$. This allows us to prove the following.

Lemma 2.3.11. *The image of* Ψ_n *is*

$$\Psi_n(\mathcal{M}(n, \mathbb{H})) = \{N \in \mathcal{M}(2n, \mathbb{C}) \mid \overline{N} I_n = I_n N\}.$$

Proof. One sees that $N \in \mathcal{M}(2n, \mathbb{C})$ belongs to $\psi(\mathcal{M}(n, \mathbb{H}))$ if and only if it commutes with the endomorphism defining $\mathfrak{j}$ multiplication in $\mathbb{C}^{2n}$, i.e.

$$N \in \psi(\mathcal{M}(n, \mathbb{H})) \Leftrightarrow N \circ \bar{\ } \circ I_n = \bar{\ } \circ I_n \circ N \Leftrightarrow \bar{\ } \circ N \circ \bar{\ } \circ I_n = I_n \circ N \Leftrightarrow \ \bar{N} I_n = I_n N.$$

□

In particular, from the lemma above,

$$\det \Psi_n(M) \in \mathbb{R}_{\geq 0}$$

for $M \in \mathcal{M}(n, \mathbb{H})$. This is because $\Psi_n(M)$ is similar to its conjugate and thus complex eigenvalues of this matrix occur in pairs, counting multiplicity, with their conjugates. Coming finally to the problem of defining determinant for a quaternionic matrices let us firs note that the usual formula does not make sense at first glance due to the commutativity of multiplication. For example

$$0 = \mathfrak{k} \cdot 1 - \mathfrak{i}\mathfrak{j} = \text{"}\det\text{"} \begin{pmatrix} 1 & \mathfrak{i} \\ \mathfrak{j} & \mathfrak{k} \end{pmatrix} \neq 2\mathfrak{k} = 1 \cdot \mathfrak{k} - \mathfrak{j}\mathfrak{i} = \text{"}\det\text{"} \begin{pmatrix} 1 & \mathfrak{i} \\ \mathfrak{j} & \mathfrak{k} \end{pmatrix}.$$

Actually, there is no function on $\mathcal{M}(n, \mathbb{H})$ satisfying all the basic properties of det on $\mathcal{M}(n, \mathbb{C})$, cf. [As96]. One try is to define

$$\det M = \det \Psi_n(M)$$

for $M \in \mathcal{M}(n, \mathbb{H})$ but this is, among other features, always non-negative. This is called Study determinant of M, cf. [As96]. Another try is the so called Dieudonné determinant which turns out to be the square root of Study determinant, cf. [As96]. Both this approaches are useless for the geometric applications as they do not capture the notion of positivity of a hermitian matrix depending on the sign of the eigenvalues. For this purpose we restrict ourself in the next section to the hyperhermitian matrices.

2.4 Moore determinant of hyperhermitian matrix

We are going to define and discuss the properties of the so called Moore determinant, cf. [M22], of the hyperhermitian matrix. We take the approach via the Pfaffian of an associated complex matrices as done for example in [Dy70, Dy72]. Alternative approach is in [A03a]. For this purpose we first note the following.

Lemma 2.4.1. *For*

$$M = A + \mathfrak{j}B \in Herm(\mathbb{H}, n),$$

where $A, B \in \mathcal{M}(n, \mathbb{C})$*, the matrix*

$$\Psi_n(M)$$

is hermitian. In turn, the matrix

$$I_n\Psi_n(M)$$

is skew-symmetric.

Proof. It is easy to see that from

$$A + \mathfrak{j}B \in Herm(\mathbb{H}, n)$$

it follows

$$A + \mathfrak{j}B = \overline{(A + \mathfrak{j}B)}^T = \overline{A}^T + \overline{\mathfrak{j}B^T} = \overline{A} + \overline{B}^T\overline{\mathfrak{j}} = \overline{A}^T - \mathfrak{j}B,$$

i.e.

$$\overline{A}^T = A, B^T = -B.$$

The statements follow from that easily, by a direct computation due to the formula I.2.2. □

Having this we make the definition of the Moore determinant as follows.

Definition 2.4.2. *For* $M \in Herm(\mathbb{H}, n)$ *we define the Moore determinant of* M*, still denoted by* $\det M$*, as*

$$\det M = Pf(I_n)Pf(I_n\Psi_n(M)). \tag{I.2.3}$$

We need the following proposition in order to show that we make no confusion by using det symbol for the Moore determinant.

Proposition 2.4.3. *For* $A \in Herm(\mathbb{C}, n)$ *its Moore determinant coincides with the usual determinant.*

Proof. For $A \in Herm(\mathbb{C}, n)$ we find that

$$I_n\Psi_n(A) = \begin{pmatrix} 0 & -\overline{A} \\ A & 0 \end{pmatrix}.$$

We have that the Moore determinant of A times the exterior product

$$e_1 \wedge ... \wedge e_{2n},$$

for e_i being a canonical basis of $\mathbb{C}^{2n}$, is

$$Pf(I_n)Pf\left(I_n\Psi_n(A)\right) e_1 \wedge ... \wedge e_{2n}.$$

Due to Definition 2.3.7 we find that this multi-vector equals

$$Pf(I_n)\frac{1}{n!}\Big(\sum_{i,j=1,...,n} -\overline{a_{ij}}e_i \wedge e_{j+n}\Big)^n$$

which equals to

$$Pf(I_n)\det\overline{(-A)}e_1 \wedge e_{1+n} \wedge ... \wedge e_n \wedge e_{n+n},$$

where det means the usual determinant. As A is hermitian the last object equals to

$$Pf(I_n)(\det A)(-1)^n e_1 \wedge e_{1+n} \wedge ... \wedge e_n \wedge e_{n+n}.$$

It is now enough to notice that this equals to

$$(\det A)e_1 \wedge ... \wedge e_{2n}$$

as we wanted. This is because due to Definition 2.3.7

$$Pf(I_n)e_1 \wedge ... \wedge e_{2n} = \frac{1}{n!}(-e_i \wedge e_{i+n})^n = (-1)^n e_1 \wedge e_{1+n} \wedge ... \wedge e_n \wedge e_{n+n}$$

which together with Theorem 2.3.6 gives

$$e_1 \wedge ... \wedge e_{2n} = Pf(I_n)(-1)^n e_1 \wedge e_{1+n} \wedge ... \wedge e_n \wedge e_{n+n}.$$

□

In the end let us recap on the eigenvalues of the quaternionic matrices as we promised in Section 2.1. As we noted there is only sense to talk about the conjugacy class of the eigenvalues of $M \in \mathcal{M}(n, \mathbb{H})$. Any complex eigenvalue of M is a complex eigenvalue of $\Psi_n(M)$, and vice versa. Since every conjugacy class of eigenvalues of M contains a complex element, sometimes even real, it corresponds to a pair of conjugated eigenvalues of $\Psi_n(M)$, in the real case to that real eigenvalue counted with multiplicity two, which are, counting multiplicities (of pairs!), at most n. This shows there are at most n conjugacy classes of eigenvalues of M or exactly n counting multiplicities (being by definition multiplicities of pairs of eigenvalues of $\Psi_n(M)$).

Let us note that for $M \in Herm(\mathbb{H}, n)$ the induced mapping of $\mathbb{C}^{2n}$ given by $\Psi_n(M)$ is hermitian. It implies that all the conjugacy classes of eigenvalues of M are real, otherwise they contain purely complex element and thus give a purely complex eigenvalue of the hermitian matrix $\Psi_n(M)$. It shows that in the hyperhermitian case there are exactly n, counting multiplicities, real eigenvalues of M being exactly the real eigenvalues of Ψ_n with their multiplicities divided by two. Having this said we state the following theorem which we will not need but it completes the picture. The proof can be found in [A03a]. A direct proof would require noting that the Pfaffian is invariant under orthonormal change of the basis.

Theorem 2.4.4. *[A03a] For any $M \in Herm(\mathbb{H}, n)$ we have*

$$\det M = \lambda_1 \cdot ... \cdot \lambda_n$$

where λ_i are eigenvalues of M listed with multiplicities, i.e.

$$\lambda_1, \lambda_1, ..., \lambda_n, \lambda_n$$

are eigenvalues of $\Psi_n(M)$ listed with multiplicities.

Chapter 3

Geometric Analysis

3.1 Hermitian geometry

This section serves mostly the purpose of collecting definitions and well known facts about differential and complex geometry. The conventions we use are fixed here as well, and this is important as some of them are against most commonly used! More than sufficient reference for the complex geometry is the introductory chapter of [H05]. The complementary source is [D12] as it takes more analytic point of view in comparison with the former one. For general differential geometry we refer to [Ta11] or the classic [KN63, KN69]. For the extended treatment of G-structures we refer to [Kob72, St83, BG08].

Let TM denote a tangent bundle of a real smooth manifold M of dimension $2n$ which is always assumed to be connected and compact in this text. Let endomorphism fields, i.e. $(1,1)$ tensor fields being by definition sections of

$$End(M) := T^*M \otimes TM,$$

act on TM **from the right**.

Definition 3.1.1. *An endomorphism field I on M is called an almost complex structure provided*

$$I^2 := I \circ I = -id_{TM}.$$

The pair (M, I) is called an almost complex manifold. When I is integrable, in the formal sense of the vanishing of the Nijenhuis tensor $[I, I]$ associated to it, we call (M, I) a complex manifold. This is equivalent, due to the Newlander-Nirenberg theorem, to the induced $Gl_n(\mathbb{C})$ structure being integrable in the strong sense i.e. to the existence of an atlas with transition functions being holomorphic.

By definition,

$$T^{\mathbb{C}}M := \mathbb{C} \otimes TM$$

is called the complexified tangent bundle. The following symbols for vector bundles will be used.

Remark 3.1.2. *We often do not distinguish between the vector bundle and the space of its, smooth if not stated otherwise, sections. This convention generally do not apply for vector fields i.e. sections of TM where we use the symbol $\Gamma(TM)$ for that space of sections.*

$$\Lambda^k(M) := \Lambda^k(T^*M)$$

$$\Lambda^k_{\mathbb{C}}(M) := \Lambda^k\left((T^{\mathbb{C}}M)^*\right) \cong \mathbb{C} \otimes \Lambda^k(M)$$

$$\Lambda^{p,q}(M) := \Lambda^{p,q}_I(M) \subset \Lambda^k_{\mathbb{C}}(M)$$

The last one denotes the bundle of complex valued forms of Hodge bidegree (p,q) with respect to the complex structure I. Let us elaborate on the last one. By definition

$$I : TM \to TM$$

but it also extends $\mathbb{C}$ linearly to

$$I : T^{\mathbb{C}}M \to T^{\mathbb{C}}M,$$

without changing the symbol denoting it. Finally, we are about to describe the action

$$I : (T^{\mathbb{C}}M)^* \to (T^{\mathbb{C}}M)^*.$$

In general there are two conventions for that action, either

$$I(\alpha)(X) = \alpha(X \circ I^{-1})$$

or

$$I(\alpha)(X) = \alpha(X \circ I),$$

for any $\alpha \in T^*M$ and $X \in \Gamma(TM)$. The first one has a notational disadvantage. If one takes, as always, $T^{1,0}M$ to be $\mathfrak{i}$ eigenspace of I acting on $T^{\mathbb{C}}M$ then in local holomorphic coordinates for I we have $T^{1,0}M = span\{\partial_{z_i}\}$ yet as one can easily check with the first convention for the I action on $(T^{\mathbb{C}}M)^*$ we have $I(dz_i) = -\mathfrak{i}dz_i$. That is the reason we choose the second convention because in this case $(\partial_{z_i})I = \mathfrak{i}\partial_{z_i}$ as well as $I(dz_i) = \mathfrak{i}dz_i$ i.e. $T^{1,0}M$ and $(T^{1,0}M)^*$ are both $\mathfrak{i}$ eigenspaces for I acting on appropriate spaces. This action can be extended to $\Lambda^k_{\mathbb{C}}(M)$ by

$$I(\alpha) = \alpha(\cdot I, ..., \cdot I)$$

for any $\alpha \in \Lambda^k_{\mathbb{C}}(M)$. We will see more important reasons for choosing the latter convention as well as having endomorphism fields acting from the right on vector and from the left on covers in the next section.

Using the Hodge decomposition and the integrability of the complex structure we obtain that on a complex manifold (M, I) the exterior differential decomposes into

$$d = \partial + \overline{\partial},$$

where the last two operators are called the Dolbeault operators and satisfy

$$\partial : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M),$$

$$\overline{\partial} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M).$$

Remark 3.1.3. *Please be warned that integrability of the complex structure is essential for this decomposition as on general almost complex manifold all we can deduce is that*

$$d = \sum_{k,l} \Pi^{k+2,l-1} \circ d \circ \Pi^{k,l} + \Pi^{k+1,l} \circ d \circ \Pi^{k,l} + \Pi^{k,l+1} \circ d \circ \Pi^{k,l} + \Pi^{k-1,l+2} \circ d \circ \Pi^{k,l},$$

where

$$\Pi^{p,q} : \oplus_{k,l=0}^{n} \Lambda^{k,l}(M) \longmapsto \Lambda^{p,g}(M)$$

is the natural projection. The summands of this decomposition are usually denoted by

$$\Theta := \sum_{k,l} \Pi^{k+2,l-1} \circ d \circ \Pi^{k,l},$$

$$\partial := \sum_{k,l} \Pi^{k+1,l} \circ d \circ \Pi^{k,l},$$

$$\overline{\partial} := \sum_{k,l} \Pi^{k,l+1} \circ d \circ \Pi^{k,l},$$

$$\bar{\Theta} := \sum_{k,l} \Pi^{k-1,l+2} \circ d \circ \Pi^{k,l}.$$

Vanishing of the Nijenhuis tensor of I is equivalent to

$$\Theta = \bar{\Theta} = 0.$$

We also introduce the twisted exterior differential

$$d^c := d^c_I = I^{-1} \circ d \circ I.$$

It is elementary to check that

$$d^c = \mathfrak{i}(\overline{\partial} - \partial)$$

and

$$dd^c + d^c d = 0,$$

the latter due to integrability of I.

Definition 3.1.4. *A Riemannian metric g on a complex manifold (M, I) is called hermitian provided*

$$g = g(\cdot I, \cdot I).$$

In that case we call (M, I, g) a hermitian manifold. As is standard to do, we associate to the hermitian metric g the so called hermitian form

$$\omega_I := \omega = g(\cdot I, \cdot).$$

It is easy to see that $\omega \in \Lambda^{1,1}(M)$.

Remark 3.1.5. *Every complex manifold M admits a compatible hermitian metric e.g.*

$$\frac{1}{2}\Big(g + g(\cdot I, \cdot I)\Big)$$

for any Riemannian metric g on M.

3.1.1 Hermitian connections and curvature

In this section we introduce some of the canonical connections on a hermitian manifold (M, I, g) and the associated notions of curvature in relation with the Chern classes. Regarding the canonical connections, they were discussed thoroughly by Gauduchon in the paper [G97a] which have become a standard reference.

Definition 3.1.6. *A connection ∇ on a hermitian manifold (M, I, g) is called hermitian provided*

$$\nabla I = \nabla g = 0.$$

Those connections in general do posses torsion since otherwise the manifold is Kähler, i.e. $d\omega = 0$, and the connection is the Levi-Civita one. In the affine space of all the hermitian connections Gauduchon has distinguished the affine line of the canonical connections denoted by ∇^t for $t \in \mathbb{R}$. Since in our considerations we will encounter only two of them we do not discuss the general context of canonical connections instead exploiting the fact that those two can be nicely characterized.

Proposition 3.1.7. *[G97a] On a hermitian manifold (M, I, g) there is a unique hermitian connection, denoted by ∇^{Ch}, which will be called the Chern connection characterized by the fact that*

$$\nabla^{0,1} = \overline{\partial}.$$

In the above, and here only, $\overline{\partial}$ denotes the Cauchy-Riemann operator of the holomorphic bundle $T^{1,0}M$, cf. [D12, H05]. The $(0, 1)$ part of the connection, cf. [D12], is obtained by projecting the $(T^{\mathbb{C}}M)^*$ component of ∇ using the Hodge decomposition on $(T^{\mathbb{C}}M)^*$. Precisely any connection

$$\nabla : \Gamma(T^{\mathbb{C}}M) \longrightarrow \Gamma\left((T^{\mathbb{C}}M)^* \otimes T^{\mathbb{C}}M\right)$$

can be decomposed as

$$\nabla = \nabla^{1,0} + \nabla^{0,1}$$

for

$$\nabla^{1,0} : \Gamma(T^{\mathbb{C}}M) \longrightarrow \Gamma\left((T^{1,0}M)^* \otimes T^{\mathbb{C}}M\right),$$
$$\nabla^{0,1} : \Gamma(T^{\mathbb{C}}M) \longrightarrow \Gamma\left((T^{0,1}M)^* \otimes T^{\mathbb{C}}M\right).$$

For the further discussion it is important that, as ∇^{Ch} is a hermitian connection, after tracing the endomorphism part of its curvature tensor $R^{\nabla^{Ch}}$ we obtain the representative of the (scaled) first Chern class in $H^2(M, \mathbb{R})$. We call this representative the Chern-Ricci curvature of g, or ω, and denote it by $Ricc(\nabla^{Ch})$. Due to the properties of this connection – vanishing of the $(1, 1)$ component of the torsion, $Ricc(\nabla^{Ch})$ is of the Hodge type $(1, 1)$. Consequently it represents the first Chern class also in the so called Bott-Chern cohomology group $H^{1,1}_{BC}(M, \mathbb{R})$, cf. [H05]. Even more importantly the following formula holds

$$Ricc(\nabla^{ch}) = tr(F^{Ch}) = -\mathrm{i}\partial\overline{\partial}log\left(\det(g_{i\bar{j}})_{i,j}\right)$$

where $g_{i\bar{j}}$ are the coefficients of g in any holomorphic chart, cf. [TW10a, TW10b]. This elementary property has serious consequences for what one has to do in order to find a metric with a given representative, say ρ, of the first Bott-Chern class $c_1^{BC}(M, I)$ as its Chern-Ricci curvature. Namely, suppose that ρ is of the form

$$\rho = -\mathrm{i}\partial\overline{\partial}log\left(\det(g_{i\bar{j}})_{i,j}\right) - \mathrm{i}\partial\overline{\partial}F.$$

Any element of $c_1^{BC}(M)$ is of this form for some smooth F if we fixed a reference metric ω. The fact that the metric $\tilde{\omega}$ has the Chern-Ricci curvature equal to ρ is equivalent to

$$-\mathrm{i}\partial\overline{\partial}log\left(\det(\tilde{g_{i\bar{j}}})_{i,j}\right) = -\mathrm{i}\partial\overline{\partial}log\left(\det(g_{i\bar{j}})_{i,j}\right) - \mathrm{i}\partial\overline{\partial}F.$$

It is elementary to observe that this is equivalent to

$$\left(\det(\tilde{g_{i\bar{j}}})_{i,j}\right) = e^{F+b}\left(\det(g_{i\bar{j}})_{i,j}\right)$$

for some $b \in \mathbb{R}$. This in turn means, by going from the chart expression to the global one, that

$$\tilde{\omega}^n = e^{F+b}\omega^n.$$

Thus prescribing the Chern-Ricci curvature for the hermitian metric is exactly the same as prescribing the volume form for it modulo a constant. It is recently an active area of study for which types of hermitian metrics it is always possible to prescribe any representative of $c_1^{BC}(M, I)$ as the Chern-Ricci curvature of the metric of that given type. As one may imagine the root of this subject is the famous Calabi conjecture which translated to the solvability of the complex Monge-Ampère equation was established originally by Yau, with many later simplifications, in [Y78]. Some of the references for this type of results are [Ch87, TW10a, TW10b, GL10, FWW10, FLY12, P15, TW17, SzTW17, P19, TW19, ChTW19]. In Chapter 7 we will make a link to this type of problems when discussing the so called quaternionic Calabi conjecture posed by Alesker and Verbitsky in [AV10].

Passing to the second announced connection for now we limit ourselves to giving the definition. We will discuss it more thoroughly in Chapter 7.

Proposition 3.1.8. *[G97a] On a hermitian manifold (M, I, g) there is a unique hermitian connection, denoted by ∇^B, which will be called the Bismut connection characterized by the fact that its torsion tensor, after lowering the upper index by g, is a three form.*

Remark 3.1.9. *This connection is named after Bismut, see [Bi89]. It was implicitly considered already in [S86] since connections with skew torsion appear naturally in its context. Because of the last fact some people insist on calling it the Strominger connection but the former name became already accepted, cf. [G97a], that is why we stick to it.*

3.2 Hyperhermitian geometry

3.2.1 Quaternionic geometries

In this section we introduce the concept of quaternionic geometric structure. Book references for this are [Be87, J00, BG08] and the papers of Salamon [S86] or Alekseevsky and Marchiafava [AM96]. The book of Boyer and Galicki contains the full list of historical references on the subject.

Probably the most natural idea to define a quaternionic manifold would be to require the existence of an atlas with transition functions having derivatives in $Gl_n\mathbb{H} \subset Gl_{4n}(\mathbb{R})$ i.e. as integrable in the strong sense $Gl_n(\mathbb{H})$ structure. This leads to a rather restrictive approach due to Proposition I from [So75] which states that functions with derivatives in $Gl_n(\mathbb{H})$ are in fact quaternionic affine transformations. This actually holds for any class of functions being holomorphic with respect to two independent complex structures. In order to broaden the class of quaternionic manifolds one can impose the following weaker requirements, which are now standard definitions yet the terminology from 70's and 80's may differ so the reader should be warned.

Definition 3.2.1. *Let M be a manifold of the real dimension $4n$ endowed with a triple of complex structures I, J, K satisfying quaternion relations. The tuple (M, I, J, K) is called a hypercomplex structure.*

Definition 3.2.2. *A manifold M of real dimension $4n$ bigger than 4 with a fixed, rank 3, subbundle $Q \subset End(TM)$ such that*

- *locally, near any point of M, Q is spanned by a triple of almost complex structures satisfying quaternion relations;*
- *there exists a torsion free connection ∇ such that $\nabla Q \subset Q$*

is called a quaternionic manifold.

Remark 3.2.3. *In the sense of the above definitions the hypercomplex and quaternionic structures are just 1 integrable $Gl_n(\mathbb{H})$ and $Gl_n(\mathbb{H}) \cdot Gl_1(\mathbb{H})$ structures respectively. By definition the latter group is $(Gl_n(\mathbb{H}) \times Sp(1))/\{(id_n, 1), (-id_n, -1)\}$. For the hypercomplex structures this follows from the theorem of Obata [Ob56]. When dropping the assumption of integrability we obtain the almost hypercomplex and almost quaternionic structures. This results in dropping the integrability condition for complex structures in Definition 3.2.1 and the second item in Definition 3.2.2. As we remarked earlier requiring the strong integrability forces the hypercomplex manifold to be a special affine manifold, cf. [So75], while the quaternionic one to be locally isomorphic to $\mathbb{HP}^n$, see Example 3.2.17, with the transition functions being projective $\mathbb{H}$ linear transformations – elements of $PGL_n(\mathbb{H})$, cf. [Ku78].*

We will be interested only in hypercomplex manifolds. We remark that quaternionic manifolds do not posses even a globally defined almost complex structure in general, cf. [GMS11], so they rather belong to the realm of real geometry, though the so called twistor space construction allows to study them via complex methods.

The announced reason for the chosen action of endomorphisms on T^*M is as follows. When considering a hypercomplex manifold (M, I, J, K) we obtain that

$$\mathbb{H} \cong \{a id_{TM} + bI + cJ + dK \mid a, b, c, d \in \mathbb{R}\} \subset End(M)$$

acts from the <u>right</u> on TM and from the <u>left</u> on T^*M. The competitive convention for the action of endomorphisms on T^*M does not give an action of $\mathbb{H}$ on T^*M at all, neither right nor left.

Remark 3.2.4. *Whenever on a hypercomplex manifold (M, I, J, K) it happens that we do not specify with respect to which complex structure the Hodge bidegree is taken it is with respect to I.*

Let (M, I, J, K) be a hypercomplex manifold. We are going to introduce the analogue of $\overline{\partial}$ operator from the complex setting. Let us remind that we have the Dolbeault operators $\partial := \partial_I$ and $\overline{\partial} := \overline{\partial_I}$ associated to the complex structure I on M. Following Verbitsky, cf. [V02], we define the differential operator ∂_J by

$$\partial_J := J^{-1} \circ \overline{\partial} \circ J. \tag{I.3.1}$$

Lemma 3.2.5. *The operator ∂_J acts on the complex forms by*

$$\partial_J : \Lambda_I^{p,q}(M) \to \Lambda_I^{p+1,q}(M).$$

Proof. Since the operator $\overline{\partial}$ acts by

$$\overline{\partial} : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p,q+1}(M)$$

it is enough to show that

$$J : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{q,p}(M)$$

because then

$$J^{-1} \circ \overline{\partial} \circ J : \Lambda_I^{p,q}(M) \to \Lambda_I^{q,p}(M) \to \Lambda_I^{q,p+1}(M) \to \Lambda_I^{p+1,q}(M).$$

For that goal it is enough to notice that

$$J : \Lambda_I^{1,0}(M) \to \Lambda_I^{0,1}(M),$$
$$J : \Lambda_I^{0,1}(M) \to \Lambda_I^{1,0}(M).$$

This is truly enough since each $\alpha \in \Lambda_I^{p,q}(M)$ is locally a sum of wedges of p elements from $\Lambda_I^{1,0}(M)$ and q elements from $\Lambda_I^{0,1}(M)$.

The two claimed properties follow from noting that $\Lambda_I^{1,0}(M)$ is an $\mathfrak{i}$ eigenspace for I acting on $T^{\mathbb{C}*}M$ and that I and J anti-commute. Explicitly, for any $\alpha \in \Lambda_I^{1,0}(M)$ we have

$$\big(I(J\alpha)\big)(X) = (J\alpha)(XI) = \alpha(XIJ) = -\alpha(XJI) = -\mathfrak{i}\alpha(XJ) = -\mathfrak{i}(J\alpha)(X)$$

so

$$J\alpha \in \Lambda_I^{0,1}(M)$$

because the elements of $\Lambda_I^{0,1}(M)$ are characterized by

$$I\beta = -\mathfrak{i}\beta.$$

Analogously we show the second claim. □

We also introduce the operator $\overline{\partial_J}$ defined formally by

$$\overline{\partial_J} := J^{-1} \circ \partial \circ J,$$

but as the operator J is real it is equal to

$$\overline{(\partial_J)}$$

as well.

It was observed by Verbitsky, [V02, V07a], that the bicomplex

$$\Big(A^{p,q} := \Lambda_I^{p+q,0}(M), \partial, \partial_J\Big),$$

called by him the quaternionic Dolbeault bicomplex, constitutes an analogue of the Dolbeault bicomplex

$$\Big(\Lambda^{p,q}, \partial, \overline{\partial}\Big)$$

from the complex case. More importantly he proved, [V07a], that it is isomorphic to the Hodge decomposition of the so called Salamon complex [S86], introduced originally in the broader context of quaternionic manifolds. It is elementary to check the following properties of the operators ∂, $\overline{\partial}$, ∂_J and $\overline{\partial_J}$, proving in particular that $\big(\Lambda_I^{p+q,0}(M), \partial, \partial_J\big)$ is truly a bicomplex.

Lemma 3.2.6. *For a hypercomplex manifold (M, I, J, K) the following hold*

$$\partial_0\partial_1 + \partial_1\partial_0 = 0,$$

$$\partial_0 \textit{ satisfies the Leibnitz rule,}$$

for any ∂_0*,* $\partial_1 \in \{\partial, \overline{\partial}, \partial_J, \overline{\partial_J}\}$.

Proof. We just would like to remark that checking the first property, which is the only one that may cause troubles at all, is easier than proposed originally by Verbitsky in [V02] for the particular pair of ∂ and ∂_J. One can simply use the facts that

$$dd^c_J + d^c_J d = 0, \tag{*}$$

$$dd^c_K + d^c_K d = 0, \tag{**}$$

what follows from the integrability of J and K as we have seen in the previous section. Then rewriting d as $\partial + \overline{\partial}$ we obtain

$$d = \partial + \overline{\partial},$$

$$d^c_I = \mathfrak{i}(\overline{\partial} - \partial),$$

$$d^c_J = J^{-1} \circ (\partial + \overline{\partial}) \circ J = \overline{\partial_J} - \partial_J,$$

$$d^c_K = (IJ)^{-1} \circ d \circ (IJ) = J^{-1} \circ (I^{-1} \circ d \circ I) \circ J = J^{-1} \circ d^c_I \circ J = \mathfrak{i}(\partial_J - \overline{\partial_J}).$$

Comparing both sides in (*) and (**) and taking into an account the Hodge bidegrees with respect to I gives the claim. More precisely (*) gives

$$\partial\overline{\partial_J} + \partial\partial_J + \overline{\partial\partial_J} + \overline{\partial}\partial_J + \overline{\partial_J}\partial + \partial_J\partial + \overline{\partial_J\partial} + \partial_J\overline{\partial} = 0, \tag{A}$$

while (**) gives

$$\partial\partial_J - \partial\overline{\partial_J} + \overline{\partial}\partial_J - \overline{\partial\partial_J} + \partial_J\partial - \overline{\partial_J}\partial + \partial_J\overline{\partial} - \overline{\partial_J\partial} = 0. \tag{B}$$

It turns out that

$$\partial\partial_J + \partial_J\partial = 0,$$

as it is the component of bidegree $(2, 0)$ of the left hand side of (A),

$$\overline{\partial\partial_J} + \overline{\partial_J\partial} = 0,$$

as it is of bidegree $(0, 2)$ of (A),

$$\partial\overline{\partial_J} + \overline{\partial}\partial_J + \overline{\partial_J}\partial + \partial_J\overline{\partial} = 0, \tag{C}$$

as it is of bidegree $(1, 1)$ of (A),

$$-\partial\overline{\partial_J} + \overline{\partial}\partial_J - \overline{\partial_J}\partial + \partial_J\overline{\partial} = 0, \tag{D}$$

as it is of bidegree $(1, 1)$ of (B). Adding and subtracting (C) and (D) we obtain

$$\overline{\partial}\partial_J + \partial_J\overline{\partial} = 0$$

and

$$\partial\overline{\partial_J} + \overline{\partial_J}\partial = 0.$$

The only two remaining identities

$$\partial\overline{\partial} + \overline{\partial}\partial = 0,$$

$$\partial_J\overline{\partial_J} + \overline{\partial_J}\partial_J = J^{-1} \circ (\overline{\partial}\partial + \partial\overline{\partial}) \circ J = 0$$

of course do hold. □

For the further use we would like to introduce also the analogue of the real forms in the complex setting. This is done as follows and is based on the exposition in [V02, AV06]. As on any complexified space, we have the bar operator $\overline{}$ defined on complex forms on M. It acts by

$$\overline{} : \Lambda_I^{q,p}(M) \to \Lambda_I^{p,q}(M).$$

Composing it with the J operator we obtain

$$\mathcal{J} := \overline{} \circ J : \Lambda_I^{p,q}(M) \to \Lambda_I^{p,q}(M).$$

Note that $\overline{}$ and J commute since J is a real operator. Consequently

$$\mathcal{J}^2 = (-1)^{p+q} id_{\Lambda_I^{p,q}(M)}.$$

This justifies the following.

Lemma 3.2.7. *On the hypercomplex manifold (M, I, J, K), for any p and q such that $p+q$ is even the operator $\mathcal{J}$ is an involution and consequently has only the eigenvalues* 1 *and* -1.

Definition 3.2.8. *In the situation from the lemma above we denote the* 1 *eigenspace of $\mathcal{J}$ by*

$$\Lambda_{I,\mathbb{R}}^{p,q}(M) := \left\{ \frac{1}{2}\big(\alpha + \mathcal{J}(\alpha)\big) \,\Big|\, \alpha \in \Lambda_I^{p,q}(M) \right\}$$

and call this a space of q-real (p, q) forms. We will be primarily interested in the case when $p = 2k$ and $q = 0$.

Proposition 3.2.9. *[AV06] Let (M, I, J, K) be a hypercomplex manifold and $f : M \to \mathbb{R}$ a smooth real function, then*

$$\partial_J f = J^{-1}\overline{\partial} f,$$
$$\partial\partial_J f \in \Lambda_{I,\mathbb{R}}^{2,0}(M).$$

Definition 3.2.10. *When we specify a Riemannian metric g on a hypercomplex manifold (M, I, J, K), hermitian with respect to I, J and K, what is equivalent to being hermitian with respect to the whole sphere*

$$S_M^2 := \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$$

of complex structures on M, then (M, I, J, K, g) is called a hyperhermitian manifold. For a hyperhermitian (M, I, J, K, g) and any $L \in S_M^2$ we take the associated hermitian form to be

$$\omega_L(X, Y) = g(XL, Y)$$

for any $X, Y \in \Gamma(TM)$. We define also the holomorphic symplectic form

$$\Omega = \omega_J - \mathrm{i}\omega_K$$

associated to g.

Remark 3.2.11. *As we will see in the chapter on HKT metrics, Chapter 7, Ω will play the role of ω in hermitian geometry and the name will become clear there as well. It is elementary to check that $\Omega \in \Lambda_{I,\mathbb{R}}^{2,0}(M)$.*

Remark 3.2.12. *As in the complex case, every hypercomplex manifold M admits a compatible hyperhermitian metric, e.g.*

$$\frac{1}{4}\Big(g + g(\cdot I, \cdot I) + g(\cdot J, \cdot J) + g(\cdot K, \cdot K)\Big)$$

for any Riemannian metric g on M.

We are about to present a series of simple, yet instructive, examples of hypercomplex manifolds.

Example 3.2.13. *Consider $M = \mathbb{H}^n$ as the right $\mathbb{H}$ module. We take the map*

$$\mu_n : \mathbb{H}^n \longrightarrow \mathbb{R}^{4n}$$

given by

$$\mu_n\left((q_i)_{i=0,\dots,n-1}\right) = \left((x_{4i})_{i=0,\dots,n-1}, (x_{4i+1})_{i=0,\dots,n-1}, (x_{4i+2})_{i=0,\dots,n-1}, (x_{4i+3})_{i=0,\dots,n-1}\right),$$

where

$$q_i = x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k},$$

for $i \in \{0, ..., n-1\}$, to be a global real chart. By definition ∂_{x_i} for $i \in \{0, ..., 4n-1\}$ give a trivialization of the bundle TM.

We check that $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ act on $\mathbb{H}^n$, in this coordinates, by

$$(x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k})\mathfrak{i} = -x_{4i+1} + x_{4i}\mathfrak{i} + x_{4i+3}\mathfrak{j} - x_{4i+2}\mathfrak{k},$$
$$(x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k})\mathfrak{j} = -x_{4i+2} - x_{4i+3}\mathfrak{i} + x_{4i}\mathfrak{j} + x_{4i+1}\mathfrak{k},$$
$$(x_{4i} + x_{4i+1}\mathfrak{i} + x_{4i+2}\mathfrak{j} + x_{4i+3}\mathfrak{k})\mathfrak{k} = -x_{4i+3} + y_{4i+2}\mathfrak{i} - x_{4i+1}\mathfrak{j} + x_{4i}\mathfrak{k}.$$

This shows that the induced complex structures I, J, K are given in the frame $\{\partial_{x_i}\}$ by

$$(\partial_{x_{4i}})I = \partial_{x_{4i+1}}, (\partial_{x_{4i+2}})I = -\partial_{x_{4i+3}},$$

$$(\partial_{x_{4i}})J = \partial_{x_{4i+2}}, (\partial_{x_{4i+1}})J = \partial_{x_{4i+3}},$$

$$(\partial_{t_{4i}})K = \partial_{x_{4i+3}}, (\partial_{x_{4i+1}})K = -\partial_{x_{4i+2}}.$$

In this setting the map, introduced in the first chapter,

$$\phi_n : \mathbb{H}^n \ni (q_i)_{i=0,\dots,n-1} \longmapsto ((z_{2i})_{i=0,\dots,n-1}, (z_{2i+1})_{i=0,\dots,n-1}) \in \mathbb{C}^{2n},$$

where

$$q_i = z_{2i} + \mathfrak{j}z_{2i+1},$$

for $i \in \{0, ..., n-1\}$, is a global holomorphic chart for the complex structure I as an isomorphism of complex vector spaces. The relations between real and complex coordinates are

$$z_j = x_{2j} + (-1)^j x_{2j+1}\mathfrak{i},$$

for $j = 0, ..., 2n-1$. As an easy calculation shows

$$dz_{2i} = dx_{4i} + \mathfrak{i}dx_{4i+1}, dz_{2i+1} = dx_{4i+2} - \mathfrak{i}dx_{4i+3},$$

$$\partial_{z_{2i}} = \frac{1}{2}(\partial_{x_{4i}} - \mathfrak{i}\partial_{x_{4i+1}}), \partial_{z_{2i+1}} = \frac{1}{2}(\partial_{x_{4i+2}} + \mathfrak{i}\partial_{x_{4i+3}}).$$

The action of J in holomorphic coordinates for I is

$$(\partial_{z_{2i}})J = (\frac{1}{2}(\partial_{x_{4i}} - \mathfrak{i}\partial_{x_{4i+1}}))J = \frac{1}{2}(\partial_{x_{4i+2}} - \mathfrak{i}\partial_{x_{4i+3}}) = \partial_{\overline{z_{2i+1}}},$$

$$(\partial_{z_{2i+1}})J = (\frac{1}{2}(\partial_{x_{4i+2}} + \mathfrak{i}\partial_{x_{4i+3}}))J = \frac{1}{2}(-\partial_{x_{4i}} - \mathfrak{i}\partial_{x_{4i+1}}) = -\partial_{\overline{z_{2i}}},$$

$$J(dz_{2i+1}) = d\overline{z_{2i}}, J(dz_{2i}) = -d\overline{z_{2i+1}},$$

i.e.

$$(\partial_{z_j})J = (-1)^j \partial_{\overline{z_{j+(-1)^j}}} \text{ and } J(dz_j) = (-1)^{j+1} d\overline{z_{j+(-1)^j}}.$$

Take a standard, though sometimes differently normalized, Riemannian metric on $\mathbb{H}^n$

$$g = dx_{4i} \otimes dx_{4i} + dx_{4i+1} \otimes dx_{4i+1} + dx_{4i+2} \otimes dx_{4i+2} + dx_{4i+3} \otimes dx_{4i+3}.$$

We easily get the following expressions, in the introduced coordinates, for the introduced quantities associated with this hyperhermitian structure

$$\omega_I = -dx_{4i+1} \otimes dx_{4i} + dx_{4i} \otimes dx_{4i+1} + dx_{4i+3} \otimes dx_{4i+2} - dx_{4i+2} \otimes dx_{4i+3}$$

$$= dx_{4i} \wedge dx_{4i+1} + dx_{4i+3} \wedge dx_{4i+2} = \frac{\mathfrak{i}}{2}(dz_{2i} \wedge d\overline{z_{2i}} + dz_{2i+1} \wedge d\overline{z_{2i+1}}),$$

$$\omega_J = dx_{4i} \wedge dx_{4i+2} + dx_{4i+1} \wedge dx_{4i+3},$$

$$\omega_K = dx_{4i+2} \wedge dx_{4i+1} + dx_{4i} \wedge dx_{4i+3},$$

$$\Omega = \omega_J - \mathfrak{i}\omega_K = dx_{4i} \wedge dx_{4i+2} + dx_{4i+1} \wedge dx_{4i+3} - \mathfrak{i}dx_{4i+2} \wedge dx_{4i+1} - \mathfrak{i}dx_{4i} \wedge dx_{4i+3}$$

$$= (dx_{4i} + \mathfrak{i}dx_{4i+1}) \wedge (dx_{4i+2} - \mathfrak{i}dx_{4i+3}) = dz_{2i} \wedge dz_{2i+1},$$

$$h = g - \mathfrak{i}\omega_I = \Omega(\cdot, \cdot J) = dz_{2i} \otimes d\overline{z_{2i}} + dz_{2i+1} \otimes d\overline{z_{2i+1}}.$$

The last formula holds on any hyperhermitian manifold in the sense that if in a local holomorphic chart for I the metric is is given by

$$g = g^{11}_{i\bar{j}} dz_{2i} \otimes d\overline{z_{2j}} + g^{12}_{i\bar{j}} dz_{2i} \otimes d\overline{z_{2j+1}} + g^{21}_{i\bar{j}} dz_{2i+1} \otimes d\overline{z_{2j}} + g^{22}_{i\bar{j}} dz_{2i+1} \otimes d\overline{z_{2j+1}}$$

$$+ g^{11}_{\bar{i}j} d\overline{z_{2i}} \otimes dz_{2j} + g^{12}_{\bar{i}j} d\overline{z_{2i}} \otimes dz_{2j+1} + g^{21}_{\bar{i}j} d\overline{z_{2i+1}} \otimes dz_{2j} + g^{22}_{\bar{i}j} d\overline{z_{2i+1}} \otimes dz_{2j+1}$$

then

$$\Omega(\cdot, \cdot J) = 2\big(g^{11}_{i\bar{j}} dz_{2i} \otimes d\overline{z_{2j}} + g^{12}_{i\bar{j}} dz_{2i} \otimes d\overline{z_{2j+1}} + g^{21}_{i\bar{j}} dz_{2i+1} \otimes d\overline{z_{2j}} + g^{22}_{i\bar{j}} dz_{2i+1} \otimes d\overline{z_{2j+1}}\big).$$

Though it is impossible to obtain a formula for Ω not involving J since in general J does not act on I holomorphic and anti-holomorphic differentials as above in the flat case of $\mathbb{H}^n$, i.e. in general there are no quaternionic coordinates as we mentioned in the beginning.

For a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ we easily compute, for further reference,

$$\partial_J u = (J^{-1}\overline{\partial}J)u = J^{-1}(\overline{\partial}u) = J^{-1}(\sum_{j=0}^{2n-1} \partial_{\overline{z_j}} u d\overline{z_j})$$

$$= J^{-1}(\sum_{j=0}^{2n-1} \partial_{\overline{z_{j+(-1)^j}}} u d\overline{z_{j+(-1)^j}}) = \sum_{j=0}^{2n-1} \partial_{\overline{z_{j+(-1)^j}}} u (-1)^{j+1} dz_j,$$

$$\partial\partial_J u = \sum_{i,j} \left((-1)^{j+1} \partial_{z_i} \partial_{\overline{z_{j+(-1)^j}}} u \right) dz_i \wedge dz_j$$

$$= \sum_{i<j} \left((-1)^{j+1} \partial_{z_i} \partial_{\overline{z_{j+(-1)^j}}} u - (-1)^{i+1} \partial_{z_j} \partial_{\overline{z_{i+(-1)^i}}} u \right) dz_i \wedge dz_j.$$

Especially

$$\partial\partial_J \left(\frac{1}{2} \sum_{k=0}^{2n-1} z_k \overline{z_k} \right) = \sum_{k=0}^{n-1} dz_{2k} \wedge dz_{2k+1} = \Omega.$$

Example 3.2.14. *Take $\mathbb{H}$ as a hyperhermitian manifold from the previous example. Since the vector fields ∂_{x_i} are invariant with respect to the additive action of $\mathbb{H}$ on itself the hypercomplex structure as well as metric descend to the quotient of $\mathbb{H}$ by $\mathbb{Z}^4 \subset \mathbb{H}$. Topologically this is simply a four torus which turns out to be a compact hyperhermitian manifold. The difference is that this example is compact.*

Example 3.2.15. *For the next compact example take $\mathbb{H}^* := \mathbb{H} \backslash \{0\}$ with the hypercomplex structure from the first example and fix a quaternion q_0 such that $|q_0| > 1$. Then q_0 generates a subgroup of diffeomorphisms of $\mathbb{H}^*$ isomorphic to $\mathbb{Z}$ given by a left multiplication by q_0^k for $k \in \mathbb{Z}$. Take the manifold being the quotient of $\mathbb{H}^*$ by $\mathbb{Z}$. It is compact since arbitrarily big, in norm, quaternion can be divided by a power of q_0 in order to obtain something with the norm smaller than one. The complex structures I, J, K are invariant for this actions since for $L \in \{\mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$ we have*

$$[q_0^k(t + x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k})]L = q_0^k[(t + x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k})L]$$

or another words saying $q_0^k \in Gl_1(\mathbb{H})$ for any k. This is an example of the so called quaternionic Hopf manifold and the construction can be generalized to

$$(\mathbb{H}^n)^* := \mathbb{H}^{n+1} \setminus \{0\}.$$

It is interesting to know which complex Hopf manifolds admit a hypercomplex structure. This was investigated by Kato in [Ka75, Ka80] an Boyer [B88].

Example 3.2.16. *It is worth mentioning that $\mathbb{H}^*$ is interesting even before taking the quotient space since this is a Lie group with action given by quaternionic multiplication. What is more, since reals commute with any quaternion we have*

$$\mathbb{H}^* \cong \mathbb{R}^* \times S^3 \cong \mathbb{R}^* \times SU(2)$$

and so its Lie algebra is

$$\mathbb{R} \times \mathfrak{su}(2).$$

Joyce provides hypercomplex structure for this Lie algebra in his famous paper [J92] which descends to this group. Let us also note that topologically related example of

$$T^1 \times SU(2) \cong \mathbb{R}^* \times SU(2)/\mathbb{Z}$$

is another Lie group with this Lie algebra and as such it is also furnished with the hypercomplex structure provided by Joyce.

Example 3.2.17. *This is a non-example. Consider* $\mathbb{H}^{n+1} \setminus \{0\}$ *with the right action of* $\mathbb{H}^*$ *given by multiplication. The quotient space is denoted by*

$$\mathbb{HP}^n$$

and called the quaternionic projective space. This does not admit even an almost complex structure but as far as we know this is not an easy fact. For the case $n = 1$ *this follows from* $\mathbb{HP}^1 \cong S^4$ *and the latter does not posses an almost complex structure which is fairly known fact as the only spheres admitting such structures are* S^2 *and* S^6*.*

Example 3.2.18. *Finally, we collect the references for the more complicated examples. It was already proved by Boyer [B88] that in real dimension four there are not many hypercomplex manifolds after all. They are either a four torus, a K3 surface i.e. a two dimensional Calabi-Yau, or a quaternionic Hopf surface. There is a nice family of, not necessarily homogeneous, examples due to Joyce [J92]. It was proven by him that any compact Lie group after multiplying by* T^k*, where* $k \in \{1, 2, 3\}$*, if necessary, admits a hypercomplex structure. Joyce's result is strongly motivated by an earlier construction from [SSTVP88], which emerged in connection with providing examples of supersymmetric sigma models. Joyce generalized it also to principal bundles. Non-homogeneous hypercomplex structures on Joyce's homogeneous manifolds where provided also by Pedersen and Poon [PP99]. Another construction, from [J91], allows to produce a lower dimensional examples from a given hypercomplex manifold, this is the version of a standard quotient construction known for different classes of geometric structures. The so called biquotient construction performed in [J95] provides further examples. There is also a sequence of examples due to Boyer et al. on Stiefel manifolds and on* S^1 *fiber bundles over 3-Sasakian manifolds [BGM94, BGM96, BGM98]. Examples of hypercomplex structures on the so called nilmanifolds, i.e. the quotients of nilpotent groups endowed with invariant structure, were provided in [BDV09]. Further examples on fiber bundles obtained by the so called twist construction were provided by Swann in [Sw10, Sw16]. For more references to the examples one can consult [BG08].*

3.2.2 Positivity on hypercomplex manifolds

We introduce positive forms, in the quaternionic sense, on the hypercomplex manifold (M, I, J, K) – the notion of an essential importance for developing pluripotential theory associated to the quaternionic Monge-Ampère operator in $\mathbb{H}^n$, done in the next chapter, and for performing a priori estimates in the last part of the text. We follow quite closely the exposition of [AV06, V10a].

Let us start from a linear algebraic considerations. For that purpose we treat T_xM as a right $\mathbb{H}$ module, with $\mathbb{H}$ action given by I_x, J_x, K_x. Firstly, we would like to introduce the canonical orientation on the real line

$$\Lambda^{2n,0}_{I,\mathbb{R}}(T_xM).$$

Fix any $\mathbb{H}$ basis $e_1, ..., e_n$ of T_xM. This gives an isomorphism of right $\mathbb{H}$ modules

$$G : T_xM \to \mathbb{H}^n,$$

on which we use the coordinates introduced in Example 3.2.13. Consider the form

$$dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1} \in \Lambda^{2n,0}_{I,\mathbb{R}}(\mathbb{H}^n),$$

we take its pull back

$$G^*\big(dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}\big) \in \Lambda^{2n,0}_{I,\mathbb{R}}(T_xM).$$

Suppose we had chosen a different isomorphism

$$H : T_xM \to \mathbb{H}^n,$$

i.e we have chosen a different $\mathbb{H}$ basis e'_i of T_xM. We would like to show that the two induced forms differ by a positive constant.

Note that the transition map from e_i to e'_i is

$$H \circ G^{-1}.$$

Since both pulled back forms belong to $\Lambda^{2n,0}_I$ they can be written using (complex) duals e^*_i, $(e_i\mathbf{j})^*$ respectively e'^*_i, $(e'_i\mathbf{j})^*$ as

$$G^*\big(dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}\big) = e^*_0 \wedge (e_0\mathbf{j})^* \wedge ... \wedge e^*_{n-1} \wedge (e_{n-1}\mathbf{j})^*,$$

$$H^*\big(dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}\big) = e'^*_0 \wedge (e'_0\mathbf{j})^* \wedge ... \wedge e'^*_{n-1} \wedge (e'_{n-1}\mathbf{j})^*.$$

The transition matrix between those two complex bases is

$$\psi\big(H \circ G^{-1}\big)$$

as we have seen in Chapter 2. From all of these we see that the two forms differ by

$$\det\Big(\psi\big(H \circ G^{-1}\big)\Big)$$

which we have seen to be positive. We are ready to define an orientation on $\Lambda^{2n,0}_{I,\mathbb{R}}(T_xM)$.

Definition 3.2.19. *Let (M, I, J, K) be a hypercomplex manifold. The canonical orientation in $\Lambda^{2n,0}_{I,\mathbb{R}}(T_xM)$ is defined by*

$$F^*(dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1})$$

for any $\mathbb{H}$ isomorphism

$$F : T_xM \to \mathbb{H}^n.$$

We denote the closed positive half-line in $\Lambda^{2n,0}_{I,\mathbb{R}}(T_xM)$ by

$$\Lambda^{2n,0}_{I,\mathbb{R}\ \geq 0}(T_xM)$$

and call its elements strongly positive.

The space of strongly positive elements in $\Lambda^{2k,0}_{I,\mathbb{R}}(T_xM)$, denoted by

$$SP^{2k}(T_xM),$$

is defined as these forms which are convex combinations of the forms

$$F^*(\alpha) \text{ for an } \mathbb{H} \text{ linear } F : T_xM \to \mathbb{H}^k \text{ and } \alpha \in \Lambda^{2k,0}_{I,\mathbb{R}\ \geq 0}(\mathbb{H}^k).$$

An element $\alpha \in \Lambda^{2k,0}_{I,\mathbb{R}}(T_xM)$ *is called weakly positive if*

$$\alpha \wedge \beta \in \Lambda^{2n,0}_{I,\mathbb{R}\ \geq 0}(T_xM)$$

for any strongly positive $\beta \in SP^{2(n-k)}(T_xM)$. *We write*

$$\alpha \in \Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(T_xM).$$

We define two types of cones

$$\Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(M) \subset \Lambda^{2k,0}_{I,\mathbb{R}}(M),$$

$$SP^{2k}(M) \subset \Lambda^{2k,0}_{I,\mathbb{R}}(M),$$

as these forms which belong pointwise to $\Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(T_xM)$, *respectively* $SP^{2k}(T_xM)$.

Remark 3.2.20. *In this text we will write* $\alpha \geq 0$ *for a weakly positive forms and call such forms positive ones.*

The following proposition contains basic properties of the forms pf the types introduced above. The proofs are standard and we refer to [AV06, V10a] for them.

Proposition 3.2.21. *[AV06] Let* (M, I, J, K) *be a compact hypercomplex manifold. Then* $SP^{2k}(M)$ *and* $\Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(M)$ *are closed, convex cones with nonempty interior. Moreover the following inclusions hold*

$$SP^{2k}(M) \subset \Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(M),$$

$$SP^{2k}(M) \wedge SP^{2}(M) \subset SP^{2(k+1)}(M),$$

$$\Lambda^{2k,0}_{I}(T_xM) = span_{\mathbb{C}} SP^{2k}(T_xM),$$

$$SP^{2k}(M) = \Lambda^{2k,0}_{I,\mathbb{R}_{\geq 0}}(M), \ \textit{for } k = 0, 1, n-1, n,$$

$$\Lambda^{2,0}_{I,\mathbb{R}_{\geq 0}}(M) = \left\{\eta \in \Lambda^{2,0}_{I,\mathbb{R}}(M) \,\middle|\, \eta(X, XJ) \geq 0 \textit{ for all } X \in \Gamma(TM)\right\}$$

$$= \left\{\eta \in \Lambda^{2,0}_{I,\mathbb{R}}(M) \,\middle|\, \eta(Z, \overline{Z}J) \geq 0 \textit{ for all } Z \in \Gamma(T^{1,0}M)\right\}.$$

For further discussion let us also introduce the bundle $S_{\mathbb{H}}(M)$ of (real parts of) hyperhermitian forms on TM. Explicitly this is a subbundle of $T^*M \otimes T^*M$ such that the fiber over $x \in M$ consists of the elements being symmetric and I_x, J_x, K_x invariant. The following lemma translates to the quaternionic situation the fact that hermitian, symmetric forms correspond to real, I invariant forms on complex manifolds.

Proposition 3.2.22. *[AV06] Suppose* (M, I, J, K) *is a hypercomplex manifold. Then the map called* t *isomorphism, named after Verbitsky [V02],*

$$t : \Lambda^{2,0}_{I,\mathbb{R}}(M) \to S_{\mathbb{H}}(M)$$

is an isomorphism of vector bundles. It is given by

$$t(\eta)(X, Y) = \frac{1}{2}(\eta(X + Y, (X + Y)J) - \eta(X, XJ) - \eta(Y, YJ))$$

or simply by

$$t(\eta)(X, X) = \eta(X, XJ),$$

as the symmetric form is uniquely defined by the values on the diagonal, for any $\eta \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$ *and* $X, Y \in \Gamma(TM)$. *What is more, positive forms correspond bijectively via* t *to the non-negative definite hyperhermitian forms.*

Proof. (Sketch) Since the section of $S_{\mathbb{H}}(M)$ is uniquely defined by the values on the diagonal, it is elementary to check that for $\eta \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$ the tensor $t(\eta)$ is a section of $S_{\mathbb{H}}(M)$, i.e. it is I and J invariant and symmetric. The inverse map is given by

$$t^{-1} : S_{\mathbb{H}}(M) \ni g \longmapsto g(\cdot J, \cdot) - \mathfrak{i} g(\cdot K, \cdot).$$

□

Chapter 4

Quaternionic pluripotential theory

In this chapter we discuss the potential theory associated to the quaternionic Monge-Ampère operator in $\mathbb{H}^n$, defining of which will be the prior goal. To our knowledge, the first to consider this partial differential operator in the flat space $\mathbb{H}^n$ was Alesker, cf. [A03b, A03a]. He developed what we call the first version of pluripotential theory, corresponding to the contents of [BT76] for the complex Monge-Ampère operator. To be more precise, in [A03a] Alesker introduced the quaternionic Monge-Ampère operator as the Moore determinant of the quaternionic Hessian and the class of quaternionic plurisubharmonic functions in $\mathbb{H}^n$. He applied the procedure from [BT76] to define the quaternionic Monge-Ampère operator for continuous not necessarily smooth plurisubharmonic functions. Moreover, Alesker proved an infant version of the comparison principle, the most powerful tool in the pluripotential theory, - the minimum principle, for the continuous plurisubharmonic functions. Having this part of the theory secured Alesker solved the degenerate Dirichlet problem, i.e. for the continuous data, in [A03b]. Later, in [A05], Alesker developed the analogue of the positivity notion, introduced in the complex case by Lelong, for certain, abstractly defined, vector spaces associated to $\mathbb{H}^n$. It was later noticed by Verbitsky that the complex Alesker considers in [A05] is isomorphic, in the flat case, to the so called Salamon complex, defined for any quaternionic manifold in [S86]. Verbitsky found the complex isomorphic to the last one inside the standard De Rham complex, cf. [V02]. The positivity notion in this language, which we adopted in the last section, was presented in dept in [V10a].

In parallel, Harvey and Lawson started the project, cf. [HL09a, HL09b], lasting the recent 15 years or so on generalizing the pluripotential theory to what they call "geometric context". They took a unified approach for such theories associated to the wide class of nonlinear partial differential equations [HL09c]. Their theory covers quaternionic plurisubharmonic functions and the quaternionic Monge-Ampère equation as special cases, as we will see in the one of the following sections. This provides evidence that this are the natural notions to study. Because of the generality of their considerations the approach they take for weak solutions is the viscosity one, in contrast to ours which is distributional. Nevertheless, they showed that their solutions are distributional solutions in the case of an elliptic cone, cf. [HL09b, HL09c]. The quaternionic theory is one of those cases. Let us also note that the viscosity approach from the start restricts the singularities of the data for the Dirichlet problem associated with the operator to the continuous ones. As a consequence the results of Harvey and Lawson obtained for the quaternionic Monge-Ampère equation are exactly those of Alesker.

Finally, motivated by the paper [A12] where Alesker lays foundations for the pluripotential theory on general quaternionic manifolds, Wan started to adopt the pluripotential theory to the quaternionic Monge-Ampère operator in, what we call, the full version, as in [BT82]. The key idea is to define properly the product of currents associated to the plurisubharmonic functions. This is contained in the papers [WW17, WZ15, WK17, W17, W19, W20]. As our language is different from the one used in those papers we make a comparison, showing that this two approaches agree, in the section below. This allows us to refer to the proofs of some results to these papers instead of recreating the whole pluripotential theory to the level as in [BT82]. In our opinion this is desirable because after showing, which we do, that the quaternionic Monge-Ampère operator can be defined as a product of certain currents no original thought is needed here and the proofs reduce to changing the symbols in comparison to the, classical, complex case.

4.1 Quaternionic Monge-Ampère operator in $\mathbb{H}^n$

We would like to define the quaternionic Monge-Ampère operator as a determinant of the quaternionic Hessian. In order to define the quaternionic Hessian, for a function in $\mathbb{H}^n$, we have to introduce "the quaternionic derivatives" which formally are certain first order differential operators.

Definition 4.1.1. *For a $\mathcal{C}^2$ function $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ and any $\alpha \in \{0, ..., n-1\}$ we define the following differential operators*

$$\partial_{\bar{q_\alpha}} f = \frac{\partial}{\partial \bar{q_\alpha}} f = \frac{\partial f}{\partial \bar{q_\alpha}} = \frac{\partial f}{\partial x_{4\alpha}} + \mathfrak{i}\frac{\partial f}{\partial x_{4\alpha+1}} + \mathfrak{j}\frac{\partial f}{\partial x_{4\alpha+2}} + \mathfrak{k}\frac{\partial f}{\partial x_{4\alpha+3}}, \tag{I.4.1}$$

$$\partial_{q_\alpha} = \frac{\partial}{\partial q_\alpha} f = \frac{\partial f}{\partial q_\alpha} = \overline{\frac{\partial \bar{f}}{\partial \bar{q_\alpha}}} = \frac{\partial f}{\partial x_{4\alpha}} - \frac{\partial f}{\partial x_{4\alpha+1}}\mathfrak{i} - \frac{\partial f}{\partial x_{4\alpha+2}}\mathfrak{j} - \frac{\partial f}{\partial x_{4\alpha+3}}\mathfrak{k}. \tag{I.4.2}$$

These operators are sometimes called the Cauchy-Riemann-Fueter operators, cf. [Su79, A03a], but as noted in [A03a] they were already considered implicitly by Hamilton himself and explicitly by Maxwell.

Lemma 4.1.2. *[A03a] For any $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ of class $\mathcal{C}^2$ and any $\alpha, \beta \in \{0, ..., n-1\}$ one has*

$$\frac{\partial^2 f}{\partial \bar{q_\alpha} \partial q_\beta} := \frac{\partial}{\partial \bar{q_\alpha}} \frac{\partial}{\partial q_\beta} f = \frac{\partial}{\partial q_\beta} \frac{\partial}{\partial \bar{q_\alpha}} f =: \frac{\partial^2 f}{\partial q_\beta \partial \bar{q_\alpha}}. \tag{I.4.3}$$

*This is not true for two "holomorphic" or "anti-holomorphic" derivatives unless one takes twice the same one. Furthermore, for a **real** valued f one can show that*

$$\frac{\partial}{\partial \bar{q_\alpha}} \frac{\partial}{\partial q_\alpha} f = \frac{\partial^2 f}{\partial x_{4\alpha}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+1}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+2}^2} + \frac{\partial^2 f}{\partial x_{4\alpha+3}^2}, \tag{I.4.4}$$

$$\frac{\partial}{\partial \bar{q_\alpha}} \frac{\partial}{\partial q_\beta} f = \overline{\frac{\partial}{\partial \bar{q_\beta}} \frac{\partial}{\partial q_\alpha} f}. \tag{I.4.5}$$

Definition 4.1.3. *For a real valued function f the matrix*

$$Hess(f, \mathbb{H}) = \left(\frac{\partial^2 f}{\partial \bar{q_\alpha} \partial q_\beta} \right)_{\alpha,\beta=1,\ldots,n}$$

is called the quaternionic Hessian of f.

Remark 4.1.4. *A question may arise why not to define the quaternionic Hessian as*

$$\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q_\beta}}\right)_{\alpha,\beta=1,\dots,n} = Hess(f, \mathbb{H})?$$

The reason is that we consider $\mathbb{H}^n$ *to be the right vector space. More precisely we would like the Hessian to define the hyperhermitian form on* $\mathbb{H}^n$*, like the real Hessian defines the symmetric form on* $\mathbb{R}^n$ *and the complex one the hermitian form on* $\mathbb{C}^n$*. The hyperhermitian form, as introduced in Definition 2.2.1, is* $\mathbb{H}$ *linear on the second entry, and for the right quaternion spaces this is the only reasonable choice. Thus if we had defined the Hessian in the expected way the associated hyperhermitian form would be*

$$\bar{q_\alpha} \frac{\partial^2 f}{\partial q_\alpha \partial \bar{q_\beta}} p_\beta$$

which does not seem more natural than choosing the convention for Hessian as in Definition 4.1.3 but is not the main reason. We would expect, for any $A \in Gl_n(\mathbb{H})$*, when treating* $Hess(f, \mathbb{H})$ *as a hyperhermitian form, to have*

$$Hess(f, \mathbb{H})(Aq, Ap) = Hess(f \circ A, \mathbb{H})(q, p)$$

for any $p, q \in \mathbb{H}^n$*. This is the case only with the choice of Definition 4.1.3 as the following observation shows.*

Proposition 4.1.5. *[A03a] For* $f : \mathbb{H}^n \longrightarrow \mathbb{H}$ *a* $\mathcal{C}^2$ *function,* $A \in Gl_n(\mathbb{H})$ *and*

$$\tilde{f} = f \circ A : \mathbb{H}^n \longrightarrow \mathbb{H}$$

we have

$$Hess(\tilde{f})(r) = A^* Hess(f)(Ar)A$$

for all $r \in \mathbb{H}^n$*.*

Following the original approach of Alesker [A03a, A03b], we define the quaternionic Monge-Ampère operator in $\mathbb{H}^n$. This is motivated by the real and complex counterparts.

Definition 4.1.6. *The action of the quaternionic Monge-Ampère operator in* $\mathbb{H}^n$ *on a* $\mathcal{C}^2$ *function* $f : \mathbb{H}^n \longrightarrow \mathbb{R}$ *is given by*

$$MA_{\mathbb{H}}(u) := \det\big(Hess(f, \mathbb{H})\big)$$

where the determinant means the Moore determinant, cf. Definition 2.4.2.

Remark 4.1.7. *Let us note that, by overlooking the observation from the previous remark, Alesker defined the Monge-Ampère operator in [A03a] as*

$$\det\left(\frac{\partial^2 f}{\partial q_\alpha \partial \bar{q_\beta}}\right)_{\alpha,\beta=1,\dots,n}.$$

This is the same operator. This follows from the fact that the Moore determinant of the transposed hyperhermitian matrix is the same, as one can verify. This requires though noting this fact constantly in some proofs, whenever Proposition 4.1.5 is used.

We would like to make a simple, but fundamental, observation made already in [A05, AV06] but without exploiting its consequences. The operator defined above can be expressed in terms of product of certain differential forms in $\mathbb{H}^n$. This trivial observation allows one to repeat the whole pluripotential theory as developed by Bedford and Taylor in [BT82] for the complex Monge-Ampère operator. As we mentioned above, Alesker did this in [A03a, A03b] but the methods were based on those from [BT76]. Let us remind that in complex case

$$\det\Big(Hess(u,\mathbb{C})\Big)(\mathfrak{i}dz_1\wedge d\overline{z_1})\wedge\ldots\wedge(\mathfrak{i}dz_n\wedge d\overline{z_n})=\frac{1}{n!}(\mathfrak{i}\partial\overline{\partial}u)^n$$

for any smooth $u:\mathbb{C}^n\to\mathbb{R}$, where

$$Hess(u,\mathbb{C})=\big(\partial_{z_i}\partial_{\overline{z_j}}u\big)_{i,j\in\{0,\ldots,n-1\}}$$

is by definition the complex Hessian of u. We are about to prove the same for the quaternionic Hessian and the Moore determinant. As we mentioned this was originally checked by Alesker and Verbitsky and in a different language by Wan in [WW17]. As our definition of the Moore determinant is through Pfaffian we can not refer to those proofs and for the convenience of the reader we perform it.

Proposition 4.1.8. *[AV06] For a smooth function $u:\mathbb{H}^n\to\mathbb{R}$ we have*

$$(\partial\partial_J u)^n=\frac{n!}{4^n}\det\left(\frac{\partial^2 u}{\partial_{\overline{q_l}}\partial_{q_k}}\right)_{l,k\in\{0,\ldots,n-1\}}dz_0\wedge dz_1\wedge\ldots\wedge dz_{2n-2}\wedge dz_{2n-1}.$$

Proof. We use the coordinate systems ϕ_n, giving an isomorphism between $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ as a complex vector spaces, and μ_n from Example 3.2.13. We also use the formulas obtained there. In particular we know that

$$\partial\partial_J u=\sum_{i,j}\Big((-1)^{j+1}\partial_{z_i}\partial_{\overline{z_{j+(-1)^j}}}u\Big)\,dz_i\wedge dz_j$$

$$=\sum_{i<j}\Big((-1)^{j+1}\partial_{z_i}\partial_{\overline{z_{j+(-1)^j}}}u-(-1)^{i+1}\partial_{z_j}\partial_{\overline{z_{i+(-1)^i}}}u\Big)\,dz_i\wedge dz_j$$

$$=\sum_{l<k}\big(\partial_{z_{2k}}\partial_{\overline{z_{2l+1}}}u-\partial_{z_{2l}}\partial_{\overline{z_{2k+1}}}u\big)\,dz_{2l}\wedge dz_{2k}$$

$$+\sum_{l,k}\big(\partial_{z_{2l}}\partial_{\overline{z_{2k}}}u+\partial_{z_{2k+1}}\partial_{\overline{z_{2l+1}}}u\big)\,dz_{2l}\wedge dz_{2k+1}$$

$$+\sum_{l<k}\big(\partial_{z_{2l+1}}\partial_{\overline{z_{2k}}}u-\partial_{z_{2k+1}}\partial_{\overline{z_{2l}}}u\big)\,dz_{2l+1}\wedge dz_{2k+1}.$$

Furthermore, we note that for any smooth function $u:\mathbb{H}^n\to\mathbb{H}$ and any $l\in\{0,\ldots,n-1\}$

$$\partial_{\overline{q_l}}u=\partial_{x_{4l}}u+\mathfrak{i}\partial_{x_{4l+1}}u+\mathfrak{j}\partial_{x_{4l+2}}u+\mathfrak{k}\partial_{x_{4l+3}}u=2\big(\partial_{\overline{z_{2l}}}u+\mathfrak{j}\partial_{\overline{z_{2l+1}}}u\big),$$

as well as for any smooth function $u:\mathbb{H}^n\to\mathbb{R}$ and any $k\in\{0,\ldots,n-1\}$

$$\partial_{q_k}u=\partial_{x_{4k}}u-\mathfrak{i}\partial_{x_{4k+1}}u-\mathfrak{j}\partial_{x_{4k+2}}u-\mathfrak{k}\partial_{x_{4k+3}}u=2\big(\partial_{z_{2k}}u-\mathfrak{j}\partial_{\overline{z_{2k+1}}}u\big).$$

From this we see the following relation between complex and quaternionic Hessians.

Lemma 4.1.9. *For a smooth function $u : \mathbb{H}^n \to \mathbb{R}$ and any $l, k \in \{0, ..., n-1\}$*

$$\partial_{\overline{q_l}}\partial_{q_k} u$$
$$= \left(2\partial_{\overline{z_{2l}}} + 2\mathfrak{j}\partial_{\overline{z_{2l+1}}}\right)\left(2\partial_{z_{2k}} u - 2\mathfrak{j}\partial_{\overline{z_{2k+1}}} u\right)$$
$$= 4\left(\partial_{\overline{z_{2l}}}\partial_{z_{2k}} u + \partial_{z_{2l+1}}\partial_{\overline{z_{2k+1}}} u\right) + 4\mathfrak{j}\left(\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}} u - \partial_{z_{2l}}\partial_{\overline{z_{2k+1}}} u\right).$$

This lemma enables us to observe that

$$\Psi_n\Big(\left(\partial_{\overline{q_l}}\partial_{q_k} u\right)_{l,k}\Big)$$
$$= 4\begin{pmatrix} \left(\partial_{\overline{z_{2l}}}\partial_{z_{2k}} u + \partial_{z_{2l+1}}\partial_{\overline{z_{2k+1}}} u\right)_{l,k} & \left(-\partial_{z_{2l+1}}\partial_{\overline{z_{2k}}} u + \partial_{\overline{z_{2l}}}\partial_{z_{2k+1}} u\right)_{l,k} \\ \left(\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}} u - \partial_{z_{2l}}\partial_{\overline{z_{2k+1}}} u\right)_{l,k} & \left(\partial_{z_{2l}}\partial_{\overline{z_{2k}}} u + \partial_{\overline{z_{2l+1}}}\partial_{z_{2k+1}} u\right)_{l,k} \end{pmatrix}.$$

This consequently means that

$$I_n\Psi_n\Big(\left(\partial_{\overline{q_l}}\partial_{q_k} u\right)_{l,k}\Big)$$
$$= 4\begin{pmatrix} -\left(\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}} u - \partial_{z_{2l}}\partial_{\overline{z_{2k+1}}} u\right)_{l,k} & -\left(\partial_{z_{2l}}\partial_{\overline{z_{2k}}} u + \partial_{\overline{z_{2l+1}}}\partial_{z_{2k+1}} u\right)_{l,k} \\ \left(\partial_{\overline{z_{2l}}}\partial_{z_{2k}} u + \partial_{z_{2l+1}}\partial_{\overline{z_{2k+1}}} u\right)_{l,k} & \left(-\partial_{z_{2l+1}}\partial_{\overline{z_{2k}}} u + \partial_{\overline{z_{2l}}}\partial_{z_{2k+1}} u\right)_{l,k} \end{pmatrix}.$$

From all of this and Definition 2.3.7 we see, by taking $e_{i+1} := dz_{2i}$ and $e_{i+1+n} := dz_{2i+1}$, that

$$\frac{4^n}{n!}\left(\partial\partial_J u\right)^n$$
$$= Pf\Big(-I_n\Psi_n\big(Hess(u, \mathbb{H})\big)\Big) dz_0 \wedge dz_2 \wedge ...dz_{2n-2} \wedge dz_1 \wedge dz_3 \wedge ... \wedge dz_{2n-1}$$
$$= (-1)^n Pf\Big(I_n\Psi_n\big(Hess(u, \mathbb{H})\big)\Big) dz_0 \wedge dz_2 \wedge ...dz_{2n-2} \wedge dz_1 \wedge dz_3 \wedge ... \wedge dz_{2n-1}.$$

On the other hand

$$\det\left(\frac{\partial^2 u}{\partial_{\overline{q_l}}\partial_{q_k}}\right)_{l,k\in\{0,...,n-1\}} dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}$$
$$= Pf(I_n) Pf\Big(I_n\Psi_n\big(Hess(u, \mathbb{H})\big)\Big) dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}.$$

The proof reduces thus to showing

$$(-1)^n dz_0 \wedge dz_2 \wedge ... \wedge dz_{2n-2} \wedge dz_1 \wedge dz_3 \wedge ... \wedge dz_{2n-1}$$
$$= Pf(I_n) dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}.$$

This is easy to verify by rewriting in the basis e_s as

$$(-1)^n e_1 \wedge ... \wedge e_{2n} = Pf(I_n) e_1 \wedge e_{1+n} \wedge ... \wedge e_n \wedge e_{n+n}$$

which was shown in the last line of the proof of Proposition 2.4.3. □

Remark 4.1.10. *It is trivial to check that for a smooth real valued function u we have*

$$(dd^c u)^{2n} = 2^{2n}(\mathfrak{i}\partial\overline{\partial} u)^{2n}$$
$$= 4^{2n}(2n)! \det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}\right)\left(\frac{\mathfrak{i}}{2} dz_0 \wedge d\overline{z_0}\right) \wedge ... \wedge \left(\frac{\mathfrak{i}}{2} dz_{2n-1} \wedge d\overline{z_{2n-1}}\right).$$

4.2 Quaternionic plurisubharmonic functions

In this section we define the quaternionic plurisubharmonic functions in $\mathbb{H}^n$, the exposition is based on [A03a, A03b, WK17]. They were introduced in the first reference but independently by G. Henkin as Alesker notes there. One should note that this class of functions was also, implicitly, discussed by Hörmander in his book [Hö07] as a special case, corresponding to the group $GL_n(\mathbb{H}) \cdot Sp(1)$ – his considerations were more general, cf. Chapters V and VI.

Definition 4.2.1. *Let D be a domain in $\mathbb{H}^n$. We call an upper semi-continuous function $f : D \to \mathbb{R}$ (strictly) quaternionic plurisubharmonic (abbreviated qpsh) if f restricted to any affine right quaternionic line intersected with D is (strictly) subharmonic as a function on a domain in $\mathbb{R}^4$. The set of all qpsh functions on D is denoted by $\mathcal{QPSH}(D)$.*

Both assertions from the following remark are easy consequences of Fubini's theorem and the fact that a quaternionic line is a complex two plane.

Remark 4.2.2. *A qpsh function f is subharmonic as a function from a domain in $\mathbb{R}^{4n}$. If we fix $\mathfrak{t} \in \{a\mathfrak{i} + b\mathfrak{j} + c\mathfrak{k} \mid a^2 + b^2 + c^2 = 1\}$ an imaginary unit and consider $\mathbb{H}^n$ as a complex vector space where multiplication by $\mathfrak{i}$ is given by the right multiplication by $\mathfrak{t}$ then plurisubharmonic functions with respect to this complex structure are qpsh. We will use that remark only for $\mathfrak{t} = \mathfrak{i}$ i.e. only for $\mathbb{H}^n$ treated as $\mathbb{C}^{2n}$ via the chart ϕ_n.*

For a smooth function plurisubharmonicity is equivalent to the non-negativity of the quaternionic Hessian, being by definition the non-negativity of the associated hyperhermitian form.

Proposition 4.2.3. *[A03a] A function $f : D \to \mathbb{R}$ of class $\mathcal{C}^2$ is qpsh if and only if*

$$Hess(f, \mathbb{H}) = \left(\frac{\partial^2 f}{\partial \bar{q_\alpha} \partial q_\beta} \right)_{\alpha,\beta=1,\ldots,n}$$

is non-negative definite, which by definition means that

$$\overline{p_\alpha} \frac{\partial^2 f}{\partial \bar{q_\alpha} \partial q_\beta} p_\beta \geq 0$$

for any $p \in \mathbb{H}^n$.

Most of the standard properties of plurisubharmonic functions, like approximation by smooth ones, mean value inequalities, convergence theorems, balayage procedure etc., see [D12, GZ17, K05, Hö07] for the panoramic view, still do hold for this class of functions. We will provide a precise reference once a certain property is used. There are differences though as well, mostly coming from an analytic point of view. Some of them will be discussed in the last section of this chapter. For the purpose of the next section we need the following.

Proposition 4.2.4. *A smooth function $f : D \to \mathbb{R}$ is qpsh if and only if $\partial\partial_J f \geq 0$.*

Proof. This follows from Proposition 3.2.22 and an easy computation, based on the one performed in the proof of Proposition 4.1.8,

$$t(\partial\partial_J f)\big(x_{4i}\partial_{x_{4i}} + x_{4i+1}\partial_{x_{4i+1}} + x_{4i+2}\partial_{x_{4i+2}} + x_{4i+3}\partial_{x_{4i+3}},$$

$$x_{4i}\partial_{x_{4i}} + x_{4i+1}\partial_{x_{4i+1}} + x_{4i+2}\partial_{x_{4i+2}} + x_{4i+3}\partial_{x_{4i+3}})$$

$$= \frac{1}{4}(x_{4i} - x_{4i+1}\mathfrak{i} - x_{4i+2}\mathfrak{j} - x_{4i+3}\mathfrak{k}) \left(\frac{\partial^2 f}{\partial_{\overline{q_i}}\partial_{q_j}}\right) (x_{4j} + x_{4j+1}\mathfrak{i} + x_{4j+2}\mathfrak{j} + x_{4j+3}\mathfrak{k})$$

for any $x_l \in \mathbb{R}$ and $l \in \{0, ..., 4n-1\}$. □

4.3 Potential theory associated to quaternionic Monge-Ampère operator

This is the core section of this chapter. Basic references for quaternionic pluripotential theory are [AV06, WZ15, WK17, WW17]. For the further usage we introduce in $\mathbb{H}^n$ the differential forms

$$\omega = \sum_{i=0}^{2n-1} \frac{\mathfrak{i}}{2} dz_i \wedge d\overline{z_i},$$

$$\omega_{2n} := \frac{\omega^{2n}}{(2n)!} = \frac{1}{4^n}\Omega_n \wedge \overline{\Omega_n},$$

$$\Omega = \sum_{i=0}^{n-1} dz_{2i} \wedge dz_{2i+1},$$

$$\Omega_n := \frac{\Omega^n}{n!} = dz_0 \wedge dz_1 \wedge ... \wedge dz_{2n-2} \wedge dz_{2n-1}.$$

Since we will extensively use facts from pluripotential theory, reproved in the quaternionic setting by Wan, Wang, Kang and Zhang it is desirable to compare differential operators ∂, ∂_J which we use with their formally defined operators d_0, d_1. Those were introduced by Wan and Wang in [WW17] to which we refer for more details. They consider the following "coordinates" given in our coordinates from Example 3.2.13 by

$$z^{j0} = x_{2j} + (-1)^{j+1} x_{2j+1}\mathfrak{i} = \overline{z_j},$$

$$z^{j1} = (-1)^{j+1} x_{2(j+(-1)^j)} + x_{2(j+(-1)^j)+1}\mathfrak{i} = (-1)^{j+1} z_{j+(-1)^j},$$

for $j = 0, ..., 2n-1$ and the associated formal derivatives

$$\nabla_{j0} = \partial_{x_{2j}} + (-1)^j \partial x_{2j+1}\mathfrak{i} = 2\partial_{\overline{z_j}},$$

$$\nabla_{j1} = (-1)^{j+1}\partial x_{2(j+(-1)^j)} - \partial x_{2(j+(-1)^j)+1}\mathfrak{i} = (-1)^{j+1} 2\partial_{z_{j+(-1)^j}}.$$

Afterwards they fix a complex basis

$$\omega^0, ..., \omega^{2n-1}$$

of $\mathbb{C}^{2n}$, which after choosing the canonical basis we note to be isomorphic to $(\mathbb{C}^{2n})^*$. Then they take the associated basis

$$\omega^I := \omega_{i_1} \wedge ... \wedge \omega_{i_k},$$

for $I = (i_1, ..., i_k)$ such that $i_1 < ... < i_k$ belong to $\{0, ..., 2n-1\}$, of the complex exterior product $\Lambda^k(\mathbb{C}^{2n})$. Again we note it is isomorphic to $\Lambda^k(\mathbb{C}^{2n*})$ which in turn is, only in the flat case, isomorphic to $\Lambda_I^{k,0}(\mathbb{H}^n)$. Finally, the operators defined by Wan and Wang on multivectors, after taking the mentioned isomorphisms, act between the spaces of differential forms

$$d_i : \Lambda^{k,0}_I(\mathbb{H}^n) \approx C^\infty(\mathbb{H}^n, \Lambda^k\mathbb{C}^{2n}) \to C^\infty(\mathbb{H}^n, \Lambda^{k+1}\mathbb{C}^{2n}) \approx \Lambda^{k+1,0}_I(\mathbb{H}^n),$$

for $i = 0, 1$, in the following way. Suppose that

$$F = \sum_I f_I \omega^I,$$

for the multi-index I as above, then

$$d_i F = \sum_{I, k\in\{0,\dots,2n-1\}} (\nabla_{ki} f_I)\omega^k \wedge \omega^I.$$

From the formulas for ∇_{ki}, we obtain

$$d_0 F = \sum_{I, k\in\{0,\dots,2n-1\}} (\nabla_{k0} f_I)\omega^k \wedge \omega^I = \sum_{I, k\in\{0,\dots,2n-1\}} 2\left(\partial_{\overline{z_k}} f_I\right)\omega^k \wedge \omega^I,$$

$$d_1 F = \sum_{I, k\in\{0,\dots,2n-1\}} (\nabla_{k1} f_I)\omega^k \wedge \omega^I = \sum_{I, k\in\{0,\dots,2n-1\}} 2(-1)^{k+1}\left(\partial_{z_{k+(-1)^k}} f_I\right)\omega^k \wedge \omega^I.$$

Proposition 4.3.1. *By taking the basis $\omega^k = (-1)^k dz_{k+(-1)^k}$ in the definition of d_0 and d_1 we obtain*

$$d_0 = 2\partial_J,$$

$$d_1 = -2\partial,$$

$$\Delta := d_0 d_1 = 4\partial\partial_J.$$

Proof. Let us recall that

$$\partial_J = J^{-1} \circ \overline{\partial} \circ J$$

and that J acts as

$$J(dz_k) = (-1)^{k+1} d\overline{z_{k+(-1)^k}}.$$

For $F = \sum_I f_I \omega^I$, we obtain

$$\partial F = \sum_{I, k\in\{0,\dots,2n-1\}} (\partial_{z_k} f_I) dz_k \wedge \omega^I,$$

$$\partial_J F = J^{-1} \circ \overline{\partial}\Big(\sum_I f_I J(\omega^I)\Big) = J^{-1}\left(\sum_{I, k\in\{0,\dots,2n-1\}} (\partial_{\overline{z_k}} f_I) d\overline{z_k} \wedge J(\omega^I)\right)$$

$$= J^{-1}\left(\sum_{I, k\in\{0,\dots,2n-1\}} (\partial_{\overline{z_{k+(-1)^k}}} f_I) d\overline{z_{k+(-1)^k}} \wedge J(\omega^I)\right)$$

$$= \sum_{I, k\in\{0,\dots,2n-1\}} (\partial_{\overline{z_{k+(-1)^k}}} f_I) J^{-1}(d\overline{z_{k+(-1)^k}}) \wedge \omega^I = \sum_{I, k\in\{0,\dots,2n-1\}} (\partial_{\overline{z_{k+(-1)^k}}} f_I)(-1)^{k+1} dz_k \wedge \omega^I.$$

This results in

$$d_0F = 2 \sum_{I,k\in\{0,\dots,2n-1\}} (\partial_{\overline{z_k}} f_I)\, \omega^k \wedge \omega^I$$

$$= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^k\, (\partial_{\overline{z_k}} f_I)\, dz_{k+(-1)^k} \wedge \omega^I$$

$$= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1} \left(\partial_{\overline{z_{k+(-1)^k}}} f_I\right) dz_k \wedge \omega^I = \mathbf{2}\partial_J F,$$

$$d_1F = 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1)^{k+1} \left(\partial_{z_{k+(-1)^k}} f_I\right) \omega^k \wedge \omega^I$$

$$= 2 \sum_{I,k\in\{0,\dots,2n-1\}} (-1) \left(\partial_{z_{k+(-1)^k}} f_I\right) dz_{k+(-1)^k} \wedge \omega^I = \mathbf{-2}\partial F.$$

□

Remark 4.3.2. *Let us just emphasize that the choice of ∂, ∂_J for our considerations instead of d_0, d_1 has some deeper than just conventional meaning. These are the natural intrinsic operators not only in $\mathbb{H}^n$ but on any hypercomplex manifold. In fact on an abstract hypercomplex manifold quaternionic plurisubharmonic functions are defined only with their aid, cf. [AV06], since the local chart definition is not possible due to non-integrability of a generic hypercomplex structure i.e. non-existence of quaternionic charts.*

From Proposition 4.3.1 it follows that we are able to use all results from [WZ15, WK17, WW17] as well as from [A03b, A03a, AV06]. We just give here the necessary details and refer to the mentioned papers for more of them. We implicitly assume familiarity with the distribution theory, though we fix the notation below. The excellent references for what is sufficient for us are the lecture notes of Dinh and Sibony [DS05] and the classic book of Federer, sections 4.1.1-4.1.7, [F69]. For a domain $D \subset \mathbb{H}^n$ we denote by

$\mathcal{D}^p_{[k]}(D)$ – space of complex p forms of class C^k on D with a compact support,
$\mathcal{D}^p(D)$ – space of complex p forms of class C^∞ on D with a compact support,
$\mathcal{D}^{p,q}_{[k]}(D) := \mathcal{D}^{p,q}_{I,[k]}(D)$ – space of (p,q) forms, with respect to I, of class C^k on D with a compact support,
$\mathcal{D}^{p,q}(D) := \mathcal{D}^{p,q}_I(D)$ – space of (p,q) forms, with respect to I, of class C^∞ on D with a compact support,

endowed with the usual topologies, cf. [F69], making them topological vector spaces. Their topological duals are the standard

$\mathcal{D}'_{p,[k]}(D) := \left(\mathcal{D}^{4n-p}_{[k]}(D)\right)'$ – space of currents of degree p of order k,
$\mathcal{D}'_p(D) := (\mathcal{D}^{4n-p}(D))'$ – space of currents of degree p,
$\mathcal{D}_{p,q,[k]}(D) := \left(\mathcal{D}^{2n-p,2n-q}_{[k]}(D)\right)'$ – space of currents of bidegree (p,q) of order k,
$\mathcal{D}_{p,q,[k]}(D) := (\mathcal{D}^{2n-p,2n-q}(D))'$ – space of currents of bidegree (p,q).

Remark 4.3.3. *It follows from Riesz' type theorem that currents of degree $4n$, called distributions, and of order zero are Borel measures. Those taking real values on real* *functions are called real measures and those taking non-negative values on non-negative functions – positive. For a current T being a measure we use the integration notation, i.e.*

$$T(f) := \int_D fT$$

for any test function f, and its extended action on any Borel function. We denote by $\mathcal{L}^m$ the Lebesgue measure in $\mathbb{R}^m$ normalized so that the volume of the unit cube is one.

Every element $\alpha \in \Lambda^p_{\mathbb{C}}(D)$ defines a current, still denoted by $\alpha \in \mathcal{D}'_p(D)$, whose action on a test form $\beta \in \mathcal{D}^{4n-p}(D)$ is given by

$$\alpha(\beta) := \int_D \alpha \wedge \beta.$$

The wedge product of a p current T and a q form α is defined by

$$T \wedge \alpha := T(\alpha \wedge \cdot).$$

Remark 4.3.4. *Taking the trivialization ω_{2n} of $\Lambda^{4n}_{\mathbb{C}}(D)$ and $\frac{1}{4^n}\overline{\Omega_n}$ of $\Lambda^{0,2n}(D)$ allows to treat the currents of degree 0 and bidegree $(2n, 0)$ as distributions.*

Definition 4.3.5. *[WW17] A current T of bidegree $(2p, 0)$ is called positive if for any $\alpha \in \mathcal{D}^{2n-2p,0}(D) \cap SP^{2n-2p}(D)$ the distribution*

$$T \wedge \alpha$$

is positive, i.e. for any $\alpha \in \mathcal{D}^{2n-2p,0}(D) \cap SP^{2n-2p}(D)$

$$\left(T \wedge \overline{\Omega_n}\right)(\alpha) \geq 0.$$

We already noticed, in Proposition 4.2.4, that for a smooth function being qpsh is equivalent to $\partial\partial_J u \geq 0$. Let us elaborate on this for merely upper-semicontinuous ones. It was already noted in Proposition 3.7 in [WW17], in their language, that $\partial\partial_J u$ is a positive current for any qpsh function. This follows from Proposition 4.2.4, approximation of singular qpsh functions by smooth ones and the fact that a weak limit of positive $(2, 0)$ currents is positive again. Wan and Wang showed that like in the complex case, cf. [BT82], one can define $(\partial\partial_J u)^n$ for any locally bounded u and treat it as a measure. This is possible even for some functions which are not locally bounded i.e. more singular ones but the former is already more than enough for most applications. This is done as follows. For any locally bounded u and positive current T, which is easy to show has measure coefficients, we define

$$\partial\partial_J u \wedge T := \partial\partial_J(uT).$$

The point is that this agrees with the wedge product of forms when u is smooth and that the product uT makes sense since the measurable coefficients can be multiplied by any function locally integrable with respect to this measure, in particular by a locally bounded one. We refer to Proposition 3.9 in [WW17] for details on this construction.

Remark 4.3.6. *We call*

$$(\partial\partial_J u)^n$$

the Monge-Ampère operator, the Monge-Ampère density or the Monge-Ampère mass associated to u. Taking into an account Proposition 4.1.8 this convention is compatible with Definition 4.1.6.

From there one can recreate most of the facts which hold for plurisubharmonic functions. We note that in [A03a] Alesker showed that $(\partial\partial_J u)^n$ can be interpreted as a measure for continuous u. The last result follows from the analogue of the Chern-Levine-Nirenberg inequalities, cf. Proposition 2.1.8 in [A03a] and Proposition 3.10 in [WW17]. In [WZ15] another important tool in the pluripotential theory – the quaternionic relative capacity – is introduced in the spirit of Bedford and Taylor.

Definition 4.3.7. *[WZ15] Let $K \subset D$ be a compact set, define*

$$cap(K, D) = sup\left\{\int_K (\partial\partial_J u)^n \mid u \in \mathcal{QPSH}(D),\ 0 \le u \le 1\right\}$$

to be the capacity of K associated to the quaternionic Monge-Ampère operator. If it is clear from the context with respect to which domain the capacity is measured we put

$$cap(K, D) = cap(K).$$

For a Borel set $E \subset D$ this is extended by

$$cap(E, D) = \sup\{cap(K, D) \mid K \text{ is compact in } D\}.$$

What is more authors proved the quasi-continuity of qpsh functions, cf. Theorem 1.1 in [WW17], i.e. that qpsh function are continuous off the sets of arbitrary small capacity. Most notably this allows to show the comparison principle for merely locally bounded functions and not only continuous ones, cf. Theorem 1.2 and Corollary 1.1 in [WZ15], in which case the proof is much simpler. This is probably the most powerful tool in pluripotential theory. The statement is exactly as we know it in the complex case but we recall it for reader's convenience.

Theorem 4.3.8. *[WZ15] Let $u, v \in \mathcal{QPSH}(D) \cap L^\infty_{loc}(D)$. If for any $\xi \in \partial D$,*

$$\liminf_{q \to \xi}(u(q) - v(q)) \ge 0$$

then

$$\int_{\{u<v\}} (\partial\partial_J v)^n \le \int_{\{u<v\}} (\partial\partial_J u)^n.$$

In particular if $(\partial\partial_J v)^n \ge (\partial\partial_J u)^n$ as measures then $u \ge v$ in D.

Finally, Wan and Zhang characterized maximality of a bounded qpsh function in terms of vanishing of its Monge-Ampère mass, cf. Theorem 1.3 in [WZ15]. Here we mean the maximality in the following sense.

Definition 4.3.9. *The function $u \in \mathcal{QPSH}(D)$ is maximal if for any $v \in \mathcal{QPSH}(D)$ and compact $K \subset D$ the inequality*

$$u \ge v \text{ holds on } K$$

provided the equality

$$u = v \text{ holds on } \partial K.$$

For the survey of function theoretic properties of the class $\mathcal{QPSH}(D)$ we refer to [WK17]. For the rest of the text we denote by D a fixed quaternionic strictly pseudoconvex domain defined, in the analogy with [CNS85], as follows.

Definition 4.3.10. *A smoothly bounded domain $D \subset \mathbb{H}^n$ such that there exists a domain U and the function ρ satisfying*

$$\begin{cases} \rho \in \mathcal{QPSH}(U) \cap C^2(U) \\ D \subset\subset U \\ D = \{\rho < 0\} \\ d\rho \neq 0 \ \text{on} \ \partial D \\ (\partial\partial_J \rho)^n \geq \Omega_n \ \text{on} \ U \end{cases}$$

is called quaternionic strictly pseudoconvex.

4.4 Quaternionic Monge-Ampère equation in $\mathbb{H}^n$

We state here the known results about the solvability of the Dirichlet problem for the quaternionic Monge-Ampère equation in $\mathbb{H}^n$.

In the paper [A03b] Alesker solves the problem with a smooth boundary data for the ball, applying Caffarelli, Kohn, Nirenberg and Spruck's arguments from [CKNS85]. Then using smooth solutions obtained that way, coupled with Bedford and Taylor's approach from [BT76], he solves the Dirichlet problem with the continuous boundary data and the right hand side continuous up to the boundary, when the domain is strictly pseudoconvex in quaternionic sense. The same was shown by Harvey and Lawson for the homogeneous problem, cf. Theorem 6.2 and Example 10.9 in [HL09c], and for the inhomogeneous in [HL20], cf. Theorem 2.11 and Example 6.7. But their approach was much more general in the sense that they treated a wide class of equations covering the quaternionic Monge-Ampère equation as a special case.

Theorem 4.4.1. *[A03b], [HL09c], [HL20] Let $D \subset \mathbb{H}^n$ be a quaternionic strictly pseudoconvex domain. Suppose $f \in \mathcal{C}(\overline{D})$ is non-negative and $\phi \in \mathcal{C}(\partial D)$. There exists a unique solution to the Dirichlet problem*

$$\begin{cases} \det\left(\frac{\partial^2 u}{\partial \bar{q_\alpha} \partial q_\beta}\right)_{\alpha,\beta=1,\ldots,n} = f\Omega_n \ \text{in} \ D, \\ u_{|\partial D} = \phi, \\ u \in \mathcal{C}(\overline{D}) \cap \mathcal{QPSH}(D). \end{cases}$$

Regularity of this solutions has been proved only recently by Zhu in [Z17]. The method of the paper is in turn the refinement of, now classical, methods from the paper [CNS85]. To be more precise Zhu proves the following.

Theorem 4.4.2. *[Z17] For a quaternionic strictly pseudoconvex domain D,*

$$f \in C^\infty(\overline{D} \times \mathbb{R})$$

a positive function such that f_x, where x is an $\mathbb{R}$ coordinate, is non-negative on $\overline{D} \times \mathbb{R}$ and $\phi \in C^\infty(\partial D)$ the Dirichlet problem

$$\begin{cases} \det\left(\frac{\partial^2 u(q)}{\partial \bar{q_\alpha} \partial q_\beta}\right)_{\alpha,\beta=1,\ldots,n} = f(q, \underline{u}(q)) \ \text{in} \ D, \\ u_{|\partial D} = \phi, \\ u \in \mathcal{QPSH}(D), \end{cases}$$

has a unique smooth solution.

After developing a sufficient part of the potential theory associated to the quaternionic Monge-Ampère operator Wan, motivated by the methods used originally by Cegrell and Persson [CP92], obtained a result on the solvability of the Dirichlet problem with a right hand side being a measure possessing a density with respect to the Lebesgue measure.

Theorem 4.4.3. *[W20] Suppose $D \subset \mathbb{H}^n$ is a quaternionic strictly pseudoconvex domain, $f \in L^q(D)$ for $q \geq 4$ is a non-negative function and $\phi \in C(\partial D)$. Then the Dirichlet problem*

$$\begin{cases} (\partial\partial_J u)^n = f\Omega_n \text{ in } D, \\ u_{|\partial D} = \phi, \\ u \in \mathcal{QPSH}(D) \cap C(\overline{D}), \end{cases}$$

has a unique solution.

Remark 4.4.4. *The solutions in the statement of the theorem in [W20] were claimed to be bounded but it is easy to observe that the obtained solutions, being the uniform limit of continuous functions, are continuous.*

This was the strongest existence result for the singular right hand side concerning the existence of the continuous solutions, before the paper [Sr18], leaving it open whether one can improve the exponent q. We will settle this issue in the second part of the text.

4.5 Glance from the unified point of view

Before going to the first original result of ours in the next section, we would like to leave the specialized, quaternionic, situation for the moment in order to grasp a broader perspective. This will allow us to see if it is worth to consider the issues discussed so far in this chapter.

In the recent years Harvey and Lawson started a lasting stream of research on carrying over the interplay between the classical (complex) pluripotential theory and the theory of degenerate fully nonlinear elliptic PDEs (classically the complex Monge-Ampère equation) to the broader context. This is the contents of the papers [HL09a, HL09b, HL09c, HL10, HL11, HL12, HL13, HL19, HL20]. As we will see the above discussed equation and the associated potential theory constitutes one, next to the real and the complex Monge-Ampère equations, of the three fundamental examples fitting into their theory.

We warn the reader that this section is meant to serve the motivational purpose only so will be vague from time to time as the proper presentation of the subject would require at least a separate thesis.

Let us start with the pluripotential side of the story. The adequate references are [HL09a, HL10, HL12], where the second one treats the most general cone case. Chronologically the starting point, cf. [HL09a], was in the presence of the calibration, the fundamental notion in studying special geometries, introduced by Harvey and Lawson in the celebrated paper [HL82]. To the contrary we begin with the most general case and then reduce to the described one. Suppose we are given a cone

$$\mathcal{P} \text{ in } Herm(\mathbb{R}, n) \text{ - the space of symmetric } n \times n \text{ real matrices},$$

which in case of a careful discussion has to satisfy a number of conditions. Instead of giving them we will see some (further) examples shortly, for now the reader may take

$$\mathcal{P} = \mathcal{P}^+ := \{A \in Herm(\mathbb{R}, n) \mid A \geq 0\}.$$

To any such a cone Harvey and Lawson associate a class

$$PSH(\mathcal{P})$$

of (upper semi-continuous) functions u by requiring that (in the smooth case)

$$Hess(u, \mathbb{R}) \in \mathcal{P}$$

pointwise and prove that the standard (function theoretic) results for the subharmonic functions have their analogues for those classes. More specialized situation, described with a great care in [HL12], is the case when $\mathcal{P}$ is geometrically motivated in the following sense. Suppose we are given

$$G \text{ a subset of the Grassmannian } Gr(p, \mathbb{R}^n)$$

of the p dimensional planes in $\mathbb{R}^n$. We set

$$\mathcal{P}(G)$$

to be the collection of those $A \in Herm(\mathbb{R}, n)$ which, when treated as a quadratic forms, restricted to any plane in G have the non-negative trace. After a moment of reflection one realizes that this means that the (smooth) functions belonging to

$$PSH(G) := PSH\left(\mathcal{P}(G)\right)$$

are those which are subharmonic, with respect to the flat metric in $\mathbb{R}^n$, after restricting to any plane in G. Finally, the even more specialized situation appears in the presence of the calibration Φ. Calibration is by definition a closed form in $\mathbb{R}^n$ of the comass 1, meaning that evaluated on any simple co-vector of the norm one it does not exceed one as well. In this case we define the set of the so called calibrated planes

$$G(\Phi) := \{q \in Gr(deg\Psi, \mathbb{R}^n) \mid \Phi(q) = 1\}.$$

In the above, we treat a plane as a simple co-vetor formed by taking the wedge product of any positively oriented orthonormal basis of that plane.

Having said this let us go to the mentioned three fundamental examples, those corresponding to the fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Below we always treat $\mathbb{C}^n$ and $\mathbb{H}^n$ as a real vector spaces remembering merely which planes in the associated Grassmannians correspond to the, respectively, complex and (right) quaternionic lines.

$$\mathbb{R}^n \longrightarrow G(\mathbb{R}) := Gr_{\mathbb{R}}(1, \mathbb{R}^n) \longrightarrow PSH(\mathbb{R}) := PSH\big(\mathcal{P}\left(G(\mathbb{R})\right)\big)$$
theory of convex functions

$$\mathbb{C}^n \sim \mathbb{R}^{2n} \longrightarrow G(\mathbb{C}) := Gr_{\mathbb{C}}(1, \mathbb{C}^n) \subset Gr_{\mathbb{R}}(2, \mathbb{R}^{2n}) \longrightarrow PSH(\mathbb{C}) := PSH\big(\mathcal{P}\left(G(\mathbb{C})\right)\big)$$
theory of plurisubharmonic functions

$$\mathbb{H}^n \sim \mathbb{R}^{4n} \longrightarrow G(\mathbb{H}) := Gr_{\mathbb{H}}(1, \mathbb{H}^n) \subset Gr_{\mathbb{R}}(4, \mathbb{R}^{4n}) \longrightarrow PSH(\mathbb{H}) := PSH\big(\mathcal{P}\left(G(\mathbb{H})\right)\big)$$
theory of quaternionic plurisubharmonic functions

We note that

$$\mathcal{P}(G(\mathbb{C})) = \{A \in Herm(\mathbb{R}, 2n) \mid \frac{1}{2}(A - IAI) \geq 0\},$$

$$\mathcal{P}(G(\mathbb{H})) = \{A \in Herm(\mathbb{R}, 4n) \mid \frac{1}{4}(A - IAI - JAJ - KAK) \geq 0\},$$

where I, J, and K are the matrices of the endomorphisms of $\mathbb{R}^{2n}$ and $\mathbb{R}^{4n}$ induced by the right multiplication by, respectively, $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{k}$ before forgetting this structures on $\mathbb{C}^n$ and $\mathbb{H}^n$. This shows that the (smooth) functions belonging to $PSH(\mathbb{C})$ and $PSH(\mathbb{H})$ are, respectively, those with

$$Hess(u, \mathbb{C}) = (u_{i\bar{j}})_{i,j} \geq 0 \text{ for } u_{i\bar{j}} = \tfrac{\partial^2 u}{\partial z_i \partial \overline{z_j}},$$

$$Hess(u, \mathbb{H}) = (u_{i\bar{j}})_{i,j} \geq 0 \text{ for } u_{i\bar{j}} = \tfrac{\partial^2 u}{\partial q_i \partial \overline{q_j}}.$$

This in turn follows from

$$Hess(u, \mathbb{C}) = \frac{1}{2}\left(Hess(u, \mathbb{R}) - IHess(u, \mathbb{R})I\right),$$

$$Hess(u, \mathbb{H}) = \frac{1}{4}\left(Hess(u, \mathbb{R}) - IHess(u, \mathbb{R})I - JHess(u, \mathbb{R})J - KHess(u, \mathbb{R})K\right),$$

where in the above we constantly identify the matrix from $M(n, \mathbb{K})$ with the matrix of the associated endomorphism of $\mathbb{R}^N$ via the isomorphisms introduced in Chapter 2 and Example 3.2.13.

The last, but important, remark is that this two cases i.e. of complex and quaternionic plurisubharmonic functions correspond to the calibration in the description above. The calibrations are the standard Kähler form ω_I in $\mathbb{C}^n$ and, also standard, hyperKähler form

$$\frac{1}{6}\left(\omega_I^2 + \omega_J^2 + \omega_K^2\right)$$

in $\mathbb{H}^n$. It was explained in [HL09a] that those calibrate, respectively, affine complex and quaternionic lines. This partially explains why there is such a difference between the theory of convex functions and its complex analogues. The point being that in the presence of the calibration Ψ Harvey and Lawson define the twisted differential

$$d^{\Psi},$$

which was independently discussed by Verbitsky in [V10b, V11], and what they call the Ψ Hessian

$$dd^{\Psi}.$$

The last one being a form of degree $deg\Psi$. As the reader probably expects in the described cases the complex Hessian corresponds to

$$dd^{\omega_I}$$

and the quaternionic one to

$$dd^{\frac{1}{6}\left(\omega_I^2+\omega_J^2+\omega_K^2\right)},$$

this last fact was noted in [HL09b]. We hope this discussion well motivates the study of the quaternionic pluripotential theory discussed in this chapter.

Turning to the PDE part of their development we point that it is covered by [HL09c, HL11, HL13, HL20, HL19]. For the simplicity we stick to the homogeneous case where the framework is as follows. Instead of dealing with the differential operator/equation Harvey and Lawson again fix a cone $\mathcal{P}$ as above. What is allowing them to cover an extremely wide spectrum of cases is that they treat the suitably set Dirichlet problem, Theorem 6.2 in [HL09c]. In case of the smooth functions it reduces to requiring that the Hessian of the solution belongs to the boundary of $\mathcal{P}$, i.e.

$$Hess(u, \mathbb{R}) \in \partial\mathcal{P}.$$

Of course we miss a number of key details of the discussion but the important note is, cf. Chapter 10 in [HL09c], Example 5.8 in [HL13], Examples C, D, G as well as Sections 4.5, 4.6 and 15 in [HL11], Examples 6.3, 6.4 in [HL20], that in the case of the cones as above we obtain the following operators.

$$\mathcal{P}_n(\mathbb{R}) := \mathcal{P}\left(G(\mathbb{R})\right) \longrightarrow \det Hess(u, \mathbb{R})$$
the real Monge-Ampère operator

$$\mathcal{P}_n(\mathbb{C}) := \mathcal{P}\left(G(\mathbb{C})\right) \longrightarrow \det Hess(u, \mathbb{C})$$
the complex Monge-Ampère operator

$$\mathcal{P}_n(\mathbb{H}) := \mathcal{P}\left(G(\mathbb{H})\right) \longrightarrow \det Hess(u, \mathbb{R})$$
the quaternionic Monge-Ampère operator

The last point we would like to make is that by considering a more general type of cones Harvey and Lawson covered the so called Hessian type equations over all three mentioned fields. This is done in [HL13] as follows. Let P_m be the elementary symmetric polynomial of degree m in n variables, this is an example of the so called Gårding hyperbolic polynomial. For this polynomials one can consider the associated cones

$$\mathcal{P}_m := \{x \in \mathbb{R}^n \mid P_m(x) \geq 0, ..., P_1(x) \geq 0\}.$$

Using that cones we define below the cones in $Herm(\mathbb{R}, n)$, with the convention that $\lambda(A)$ denotes the increasingly ordered n tuple of the eigenvalues of the matrix A from respectively $Herm(\mathbb{R}, n)$, $Herm(\mathbb{C}, n)$, $Herm(\mathbb{H}, n)$.

$$\mathcal{P}_m(\mathbb{R}) := \{A \in Herm(\mathbb{R}, n) \mid \lambda(A) \in \mathcal{P}_m\} \longrightarrow P_m\left(\lambda\left(Hess(u, \mathbb{R})\right)\right)$$
the real m-Hessian operator

$$\mathcal{P}_m(\mathbb{C}) := \{A \in Herm(\mathbb{R}, 2n) \mid \lambda\left(\tfrac{1}{2}\left(A - IAI\right)\right) \in \mathcal{P}_m\} \longrightarrow P_m\left(\lambda\left(Hess(u, \mathbb{C})\right)\right)$$
the complex m-Hessian operator

$$\mathcal{P}_m(\mathbb{H}) := \{A \in Herm(\mathbb{R}, 4n) \mid$$
$$\lambda\left(\tfrac{1}{4}\left(Hess(u, \mathbb{R}) - IHess(u, \mathbb{R})I - JHess(u, \mathbb{R})J - KHess(u, \mathbb{R})K\right)\right) \in \mathcal{P}_m\}$$
$$\longrightarrow P_m\left(\lambda\left(Hess(u, \mathbb{H})\right)\right)$$
the quaternionic m-Hessian operator

Those types of equations have intrigued analysts for a long time. We just mention a few results in the classical – real and complex cases. We concentrate here on the subject of our interest in this chapter – the degenerate equations. Weak solution to the real Monge-Ampère equation were found in [Al58, RT77]. An analogous result for the complex Monge-Ampère equation was obtained by Kołodziej [K95, K96, K98] generalizing prior results of Bedford and Taylor [BT76] and Cegrell and Persson [C84, CP92]. The real Hessian equations are treated in the fundamental papers [TW97, TW99, TW02]. The basic references for the complex Hessian equations are [B05] and [DK14].

4.6 Local integrability of qpsh functions

In this section we address the question of local integrability of qpsh functions in a domain $D \subset \mathbb{H}^n$ demonstrating that they exhibit an unusual property in that context. For plurisubharmonic functions in $\mathbb{C}^n$ it is known that they are locally integrable with any power $p \geq 1$. For the more general class of m-subharmonic functions in $\mathbb{C}^n$ we know that they belong to L^p_{loc} for

$$p < \frac{n}{n-m},$$

cf. [B05]. On the other hand the fundamental solution for a complex m-Hessian equation

$$f(z) = - \parallel z \parallel^{2-2\frac{n}{m}},$$

is integrable with any exponent

$$p < \frac{mn}{n-m}.$$

The conjecture of Błocki is that any m-subharmonic function is locally as integrable as the fundamental solution, cf. [B05]. This was confirmed by Dinew and Kołodziej, cf. [DK14], provided the function is bounded near the boundary of D but it still may have poles inside. In the real case the similar problem is settled. For the m-convex functions, when $m \leq \frac{n}{2}$ because otherwise they have no poles at all, it was shown in [TW99] that they belong to L^p_{loc} for

$$p < \frac{nm}{n-2m}.$$

This turns out to be optimal for the fundamental solution to the real m-Hessian equation as well. We will see that the qpsh functions are the ones which do not share this property. The proof of the proposition below is inspired by the presentation in [Hö07]. Those results are taken from [Sr18].

Proposition 4.6.1. *[Sr18] Suppose $u \in \mathcal{QPSH}(D)$ is such that $u \not\equiv -\infty$. Then*

$$u \in L^p_{loc}(D)$$

for any $p < 2$ and the bound on p is optimal. What is more if $u_j \not\equiv -\infty$ is a sequence of qpsh functions converging in $L^1_{loc}(D)$ to some u, necessarily belonging to $\mathcal{QPSH}(D)$, then the convergence holds in $L^p_{loc}(D)$ for any $p < 2$.

Proof. Suppose without loss of generality that $u \leq 0$ in a neighborhood of a quaternionic polyball $P(0,1) := \big(B(0,1)\big)^n$ of radius 1 centered at 0 contained in D, such that $u(0) >$

$-\infty$ and fix $p < 2$. Let us deal firstly with the case $n = 1$. From the Riesz representation theorem, cf. Theorem 3.3.6 in [Hö07],

$$u(q) = h(g) + \int\limits_{\|\xi\|<1} G(q,\xi)d\mu(\xi)$$

for some non positive harmonic function h in the ball $B(0,1) := B_1$, non-negative Borel measure μ and Green's function

$$G(q,\xi) = -\frac{1}{\| q-\xi \|^2} + \frac{1}{\| (q-\frac{\xi}{|\xi|^2})|\xi| \|^2}.$$

By Harnack's inequality, cf. Theorem 3.1.7 in [Hö07], for any $\| q \| < \frac{1}{2}$ we have

$$0 \le -h(q) \le \frac{1+\| q \|}{(1-\| q \|)^3}(-h(0)) \le 12(-h(0)).$$

This shows that

$$\| h \|_{L^p\left(B(0,\frac{1}{2})\right)} \le C_p |h(0)|$$

for a constant C_p, depending only on $p < 2$, which we may still need to increase (see below).

For estimating the second component of the decomposition of u let us introduce the following notation

$$H(q,\xi) = -G(q,\xi) \ge 0. \text{ For } \xi = 0 \text{ we have } H(q,0) = \tfrac{1}{\|q\|^2} - 1.$$

We consider two cases depending on whether ξ is close to the center or to the boundary of B_1.

In the first case, say when $\| \xi \| \le \frac{3}{4}$, we use the estimate

$$0 \le H(q,\xi) \le \tfrac{1}{\|q-\xi\|^2}$$

for any q and ξ, consequently

$$\left(\int\limits_{\|q\|<\frac{1}{2}} (H(q,\xi))^p \, d\mathcal{L}^4(q) \right)^{\frac{1}{p}} \le \left(\int\limits_{\|q\|<\frac{1}{2}} \frac{1}{\| q-\xi \|^{2p}} d\mathcal{L}^4(q) \right)^{\frac{1}{p}}$$

$$\le \left(\int\limits_{\|q\|<\frac{5}{4}} \frac{1}{\| q \|^{2p}} d\mathcal{L}^4(q) \right)^{\frac{1}{p}} \le C_p' \left(\frac{1}{\| \xi \|^2} - 1 \right)$$

for a constant C_p' independent of ξ and depending only on $p < 2$, since the expression $\frac{1}{\|\xi\|^2} - 1$ is bounded from below for $\| \xi \| \le \frac{3}{4}$.

In the second case, say when $\| \xi \| \ge \frac{3}{4}$, we note that for any fixed ξ the function $H(\cdot,\xi)$ is non negative and harmonic in $B_{\frac{3}{4}}$. Applying Harnack's inequality for each fixed ξ we obtain that for all $\| \xi \| \ge \frac{3}{4}$ and for all $\| q \| < \frac{3}{4}$

$$0 \le H(q,\xi) \le \left(\tfrac{3}{4}\right)^2 \tfrac{\frac{3}{4}+\|q\|}{(\frac{3}{4}-\|q\|)^3} H(0,\xi),$$

hence for all $\| \xi \| \geq \frac{3}{4}$ and $\| q \| < \frac{1}{2}$

$$0 \leq H(q,\xi) \leq 45 \left(\frac{1}{\|\xi\|^2} - 1 \right).$$

To sum up we have proven that there exists a constant $C_p = \max\{C'_p, 45\}$, independent of ξ, such that for $\| \xi \| < 1$

$$\| H(\cdot,\xi) \|_{L^p\left(B(0,\frac{1}{2})\right)} \leq C_p \left(\frac{1}{\|\xi\|^2} - 1 \right).$$

From Minkowski's inequality and Minkowski's integral inequality we obtain

$$\| u \|_{L^p\left(B(0,\frac{1}{2})\right)} \leq \| h \|_{L^p\left(B(0,\frac{1}{2})\right)} + \left(\int\limits_{\|q\|<\frac{1}{2}} \Big| \int\limits_{\|\xi\|<1} H(q,\xi) d\mu(\xi) \Big|^p d\mathcal{L}^4(q) \right)^{\frac{1}{p}}$$

$$\leq C_p|h(0)| + \int\limits_{\|\xi\|<1} \left(\int\limits_{\|q\|<\frac{1}{2}} H(q,\xi)^p d\mathcal{L}^4(q) \right)^{\frac{1}{p}} d\mu(\xi)$$

$$\leq C_p \left(|h(0)| + \int\limits_{\|\xi\|<1} \left(\frac{1}{\| \xi \|^2} - 1 \right) d\mu(\xi) \right) = C_p|u(0)|.$$

Using Fubini's theorem and the estimate above one obtains that in the case of $n \geq 1$ we have

$$\| u \|_{L^p\left(P(0,\frac{1}{2})\right)} \leq C_p^n |u(0)|.$$

To the end observe that D', the set of points in D in a neighborhood of which u is integrable with exponent p, is an open set by definition. It is closed by what we have just shown. This is so because if $q \in \overline{D'}$ and $r > 0$ is such that $P(q, 3r) \subset\subset D$ then we can find and element q' of D' within $\frac{r}{2}$ distance from q and a point $q'' \in D$ within $\frac{r}{2}$ distance from q' such that $u(q'')$ is finite. We note that $P(q'', 2r) \subset\subset D$, consequently u is integrable with the exponent p on $P(q'', r)$, and $q \in P(q'', r)$. What is more D' is nonempty by the assumption $u \not\equiv -\infty$. The bound on p is optimal as the example of $-\frac{1}{\|q_0\|^2}$ in $\mathbb{H}^n$ for $n \geq 1$ shows.

For the proof of the second assertion we note that the sequence $u_j - u$ is bounded in $L^p_{loc}(D)$ for any $1 \leq p < 2$. To prove this it is enough to show that the sequence u_j is bounded in $L^p_{loc}(D)$. Fix any point $q \in D$ and let $r > 0$ be such that $P(q, 3r) \subset\subset D$. We claim that L^p norms of u_j in $P(q, \frac{r}{2})$ are bounded. Suppose to the contrary that they are not. Let us choose a subsequence j_k such that $\| u_{j_k} \|_{L^p(P(q,\frac{r}{2}))} \longrightarrow \infty$. We know that u_{j_k}'s are locally uniformly bounded from above, cf. Theorem 3.2.13 in [Hö07], so we may assume that u and u_{j_k}'s are non positive in $P(q, 3r)$. It is possible to find a point q' within $\frac{r}{2}$ distance from q such that $u(q') > -\infty$ and $\limsup_{k\to\infty} u_{j_k}(q') = u(q')$ since both this properties hold almost everywhere in D, cf. Theorem 3.2.13 in [Hö07]. We assume w.l.o.g that $\lim_{k\to\infty} u_{j_k}(q') = u(q')$ for if not we take a subsequence again. In particular there exists $C > 0$ such that $u_{j_k}(q') > -C$ for any k. This together with the estimate we have proven shows that the sequence $\| u_{j_k} \|_{L^p(P(q',r))}$ is bounded. Because we also know that $P(q, \frac{r}{2}) \subset P(q', r)$, contradiction with $\| u_{j_k} \|_{L^p(P(q,\frac{r}{2}))} \to \infty$ is obtained. Fix $1 \leq p < 2$ and observe that for any compact $K \subset D$ we have

$$\int_K |u_j - u|^p d\mathcal{L}^{4n} = \int_K |u_j - u|^{\frac{2-p}{2}} |u_j - u|^{\frac{3p-2}{2}} d\mathcal{L}^{4n}$$

$$\leq \left(\int_K |u_j - u|^{(\frac{2-p}{2})(\frac{2}{2-p})} d\mathcal{L}^{4n} \right)^{(\frac{2-p}{2})} \left(\int_K |u_j - u|^{(\frac{3p-2}{2})(\frac{2}{p})} d\mathcal{L}^{4n} \right)^{\frac{p}{2}}$$

$$= \left(\int_K |u_j - u| d\mathcal{L}^{4n} \right)^{(\frac{2-p}{2})} \left(\int_K |u_j - u|^{(3-\frac{2}{p})} d\mathcal{L}^{4n} \right)^{\frac{p}{2}}$$

by Hölder's inequality. By the assumption the first term tends to zero while the second one is bounded since $1 \leq 3 - \frac{2}{p} < 2$. This proves that u_j tend to u in $L^p_{loc}(D)$ for any $1 \leq p < 2$. □

In contrast, the following proposition was proven in [WW17], one can verify it the way the Proposition 5.3.2 is checked.

Proposition 4.6.2. *[WW17] The function*

$$f(q) = -\frac{1}{\parallel q \parallel^2}$$

is the fundamental solution for the quaternionic Monge-Ampère operator in $\mathbb{H}^n$*. More precisely*

$$(\partial\partial_J f)^n = \frac{2^n \pi^{2n} n!}{(2n)!} \delta_0.$$

We see that the fundamental solution to the quaternionic Monge-Ampère equation is in $L^p_{loc}(\mathbb{H}^n)$ for any $p < 2n$ while a generic qpsh function only for $p < 2$ which is in contrast with the case of the mentioned classes of functions associated to the real and complex $m-$Hessian equations.

Part II

Local case

Chapter 5

Weak solutions to the Dirichlet problem

In this section we aim to solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}), \\ (\partial\partial_J u)^n = f\Omega_n \text{ in } D, \\ u_{|\partial D} = \phi, \end{cases} \tag{II.5.1}$$

where $f \in L^q(D)$ for $q > 2$, $\phi \in C(\partial D)$ and $D \subset \mathbb{H}^n$ is a smoothly bounded strictly quaternionic pseudoconvex domain, which is a global assumption for D in this part. It was done originally in [Sr18]. Let us mention that the Dirichlet problem for the complex Monge-Ampère equation with densities in L^p for $p > 1$ was solved by Kołodziej in [K96] refining the earlier result of Bedford and Taylor [BT76] for the continuous data. In fact Kołodziej proved the result for densities in appropriate Orlicz spaces being subspaces of L^1 and in particular cases reducing to L^p. For the real Monge-Ampère equation one can always solve the above problem for any density in L^1, cf. [Al58, RT77]. For the more general $m-$Hessian equations we refer to [DK14] for the complex case and to [TW99, TW02] for the real one. The sharp estimates for the exponent are known for those equations as well.

5.1 Volume–capacity comparison

The first goal is to compare complex and quaternionic Monge-Ampère operators. We start with smooth functions. In this case we have to compare the determinants of the complex and quaternionic Hessians. The idea of performing such a comparison originates with Cheng and Yau. This was noted in [CP92] where using their idea Cegrell and Persson compared the complex and real Monge-Ampère operators. Real and quaternionic Monge-Ampère operators were compared by Wan in [W20]. Let us recall that we have distinguished the set $\mathcal{PSH}(D)$ of plurisubharmonic functions in D by identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ via a chart introduced in Chapter 2.

Lemma 5.1.1. *For a function $u \in \mathcal{PSH}(D) \cap C^2(D) \subset \mathcal{QPSH}(D)$ the following holds*

$$\Big(\det(\tfrac{\partial^2 u}{\partial_{\overline{q_l}}\partial_{q_k}})\Big)^2 \geq 4^{2n} \det(\tfrac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}).$$

Proof. Let us recall that

$$Hess(u,\mathbb{C}) = \left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}\right)_{i,j=0,\dots,2n-1},$$

$$Hess(u,\mathbb{H}) = \left(\frac{\partial^2 u}{\partial_{\overline{q_l}}\partial_{q_k}}\right)_{l,k=0,\dots,n-1}.$$

Note that

$$\det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}\right)_{i,j=0,\dots,2n-1} = \det Hess(u,\mathbb{C}) = \det \overline{Hess(u,\mathbb{C})} = \det\left(\frac{\partial^2 u}{\partial_{\overline{z_i}}\partial_{z_j}}\right)_{i,j=0,\dots,2n-1}.$$

The last matrix is Hermitian positive since it is just $Hess(u,\mathbb{C})^T$. For

$$Hess(u,\mathbb{H}) = G + \mathrm{j}H$$

we defined

$$\Psi_n\left(Hess(u,\mathbb{H})\right) = \begin{pmatrix} G & -\overline{H} \\ H & \overline{G} \end{pmatrix}.$$

By Lemma 4.1.9 we obtain

$$\begin{aligned}
&\Psi_n\left(Hess(u,\mathbb{H})\right) \\
&= 4\begin{pmatrix} \left[\partial_{\overline{z_{2l}}}\partial_{z_{2k}}u + \partial_{z_{2l+1}}\partial_{\overline{z_{2k+1}}}u\right]_{l,k} & \left[-\partial_{z_{2l+1}}\partial_{\overline{z_{2k}}}u + \partial_{\overline{z_{2l}}}\partial_{z_{2k+1}}u\right]_{l,k} \\ \left[\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}}u - \partial_{z_{2l}}\partial_{\overline{z_{2k+1}}}u\right]_{l,k} & \left[\partial_{z_{2l}}\partial_{\overline{z_{2k}}}u + \partial_{\overline{z_{2l+1}}}\partial_{z_{2k+1}}u\right]_{l,k} \end{pmatrix} \\
&= 4\begin{pmatrix} \left[\partial_{\overline{z_{2l}}}\partial_{z_{2k}}u\right]_{l,k} & \left[\partial_{\overline{z_{2l}}}\partial_{z_{2k+1}}u\right]_{l,k} \\ \left[\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}}u\right]_{l,k} & \left[\partial_{\overline{z_{2l+1}}}\partial_{z_{2k+1}}u\right]_{l,k} \end{pmatrix} + 4\begin{pmatrix} \left[\partial_{z_{2l+1}}\partial_{\overline{z_{2k+1}}}u\right]_{l,k} & \left[-\partial_{z_{2l+1}}\partial_{\overline{z_{2k}}}u\right]_{l,k} \\ \left[-\partial_{z_{2l}}\partial_{\overline{z_{2k+1}}}u\right]_{l,k} & \left[\partial_{z_{2l}}\partial_{\overline{z_{2k}}}u\right]_{l,k} \end{pmatrix}.
\end{aligned}$$

Following [CP92] we introduce three matrices

$$A = \left[\partial_{\overline{z_{2l}}}\partial_{z_{2k}}u\right]_{l,k},\ B = \left[\partial_{\overline{z_{2l+1}}}\partial_{z_{2k}}u\right]_{l,k},\ C = \left[\partial_{\overline{z_{2l+1}}}\partial_{z_{2k+1}}u\right]_{l,k}.$$

Under this notation

$$\Psi_n\left(Hess(u,\mathbb{H})\right) = 4\begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix} + 4\begin{pmatrix} \overline{C} & -\overline{B} \\ -B^T & \overline{A} \end{pmatrix}.$$

Note that

$$\begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix}$$

is the conjugate of a Hessian of u with respect to the coordinates z_0, ..., z_{2n-2}, z_1, ..., z_{2n-1} so it is Hermitian positive as well. Moreover

$$\det\left(\tfrac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}\right) = \det\begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix}.$$

Consider the matrix I_n from Chapter 2

$$I_n = \begin{pmatrix} 0 & -id_n \\ id_n & 0 \end{pmatrix}$$

with the inverse

$$I_n^{-1} = \begin{pmatrix} 0 & id_n \\ -id_n & 0 \end{pmatrix}$$

and the determinant equal to one. Note that

$$I_n \begin{pmatrix} \overline{C} & -\overline{B} \\ -B^T & \overline{A} \end{pmatrix} I_n^{-1} = \begin{pmatrix} \overline{A} & B^T \\ \overline{B} & \overline{C} \end{pmatrix}.$$

The last matrix is the conjugate of the one just shown to be Hermitian positive, as such it is also Hermitian positive. Consequently

$$\begin{pmatrix} \overline{C} & -\overline{B} \\ -B^T & \overline{A} \end{pmatrix}$$

is positive as being similar to the one of that kind. Now we use the equality between Moore's determinant of a matrix M and the normalized Pfaffian of an associated complex matrix $I_n \Psi_n(M)$ due to Definition 2.4.2. In case of the original definition for the Moore's determinant this equality was proved in [Dy70]. This results in

$$\left(\det\left(\frac{\partial^2 u}{\partial_{\overline{q_l}}\partial_{q_k}}\right)\right)^2 = \det \Psi_n\big(Hess(u, \mathbb{H})\big)$$

$$= 4^{2n} \det\left(\begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix} + \begin{pmatrix} \overline{C} & -\overline{B} \\ -B^T & \overline{A} \end{pmatrix}\right)$$

$$\geq 4^{2n} \det \begin{pmatrix} A & \overline{B}^T \\ B & C \end{pmatrix} = 4^{2n} \det\left(\frac{\partial^2 u}{\partial_{z_i}\partial_{\overline{z_j}}}\right)$$

as we desired to prove. □

The announced comparison of quaternionic and complex Monge-Ampère operators, for non-smooth functions, follows from the approximation procedure.

Theorem 5.1.2. *If $u \in \mathcal{PSH}(D) \cap C(\overline{D})$ satisfy the equation*

$$(dd^c u)^{2n} = f^2 4^{2n} \omega^{2n}$$

for some non-negative $f \in L^p(D)$ and $p > 2$ then

$$(\partial\partial_J u)^n \geq f \Omega^n.$$

Proof. Since the property is local we may assume that D is strictly pseudoconvex, otherwise we argue as below but for some ball contained in D. Approximate f by a sequence of smooth positive functions f_i in L^p norm and u uniformly by a sequence of smooth functions ϕ_i on ∂D. Let us solve the family of Dirichlet problems

$$\begin{cases} u_i \in \mathcal{PSH}(D) \cap C^\infty(\overline{D}) \\ (dd^c u_i)^{2n} = f_i^2 4^{2n} \omega^{2n} \\ u_i = \phi_i \text{ on } \partial D \end{cases}$$

which is possible due to [CKNS85]. Observe that u_i converge uniformly to u due to stability of solutions for the complex Monge-Ampère equation, cf. [DK14]. From Proposition 4.1.8, Remark 4.1.10 and Lemma 5.1.1

$$(\partial\partial_J u_i)^n = \frac{1}{4^n} \det\left(\frac{\partial^2 u_i}{\partial_{\overline{q_l}}\partial_{q_k}}\right)\Omega^n \geq \sqrt{\det(\frac{\partial^2 u_i}{\partial_{z_m}\partial_{\overline{z_n}}})}\Omega^n = f_i \Omega^n$$

as measures. Right hand sides converge, as measures, to $f\Omega_n^n$ and left ones converge to $(\partial\partial_J u)^n$ since the convergence of u_i is uniform what ends the proof. □

Now we are going to prove an inequality between the Lebesgue measure and quaternionic capacity. This was an essential component of Kołodziej's proof of solvability of the complex Monge-Ampère equation for densities in appropriate Orlicz spaces, cf. [K96, K05]. Originally, i.e. in the complex setting, this inequality was proven with the usage of the other - Siciak capacity. It exploits some exceptional properties of subharmonic functions in $\mathbb{R}^2$. It is of interest to know whether one could introduce the analog of this capacity for the quaternionic Monge-Ampère operator and prove comparison applying Kołodziej's original approach. Similar inequality for the capacity associated to the complex $m-$Hessian operator was proven in [DK14] with the usage of an observation that psh functions, although being an extremal example of $m-$subharmonic ones, still realize the $m-$Hessian capacity. Here we couple that trick with the comparison of quaternionic and complex Monge-Ampère operators proved in Theorem 5.1.2.

Lemma 5.1.3. *For a fixed $p \in (1,2)$ there exists a constant $C(p,R)$ such that for any $D \subset B(0,R)$ and any Borel set $E \subset D$*

$$\mathcal{L}^{4n}(E) \leq C(p,R) cap^p(E,D).$$

Proof. We show the assertion firstly for the compact sets $K \subset D$. Suppose that $\mathcal{L}(K) \neq 0$ otherwise there is nothing to prove. Take any $\epsilon \in (0, \frac{1}{2})$ and consider

$$f = \mathcal{L}(K)^{2\epsilon - 1}\chi_K.$$

Let us solve the Dirichlet problem

$$\begin{cases} u \in \mathcal{PSH}(B) \cap C(\overline{B}) \\ (dd^c u)^{2n} = f 4^{2n}\omega^{2n} \\ u = 0 \; on \; \partial B \end{cases}$$

which is possible due to [C84]. By Theorem 5.1.2 the quaternionic Monge-Ampère operator of the solution u satisfies

$$(\partial\partial_J u)^n \geq \sqrt{f}\Omega^n.$$

Take $q = 1+\epsilon$, one checks that

$$\int_B f^q \left(4^{2n}(2n)!\right)^q d\mathcal{L}^{4n} = \left(4^{2n}(2n)!\right)^q \mathcal{L}(K)^{(2\epsilon-1)(1+\epsilon)+1}$$
$$= \left(4^{2n}(2n)!\right)^q \mathcal{L}(K)^{2\epsilon^2+\epsilon} \leq \left(4^{2n}(2n)!\right)^2 R^{4n},$$

i.e. the L^q norm of f is bounded by a quantity depending only on R. By Kołodziej's L^∞ estimate, cf. [K96, K98], there exists a constant $c(\epsilon, R)$ such that

$$\parallel u \parallel_{L^\infty(B)} \leq \frac{1}{c(\epsilon,R)}.$$

Put $v = c(\epsilon,R)u$, then since v is a qpsh function such that $-1 \leq v \leq 0$

$$cap(K,D) \geq \int_K (\partial\partial_J v)^n \geq n! c(\epsilon,R)^n \left(\mathcal{L}^{4n}(K)\right)^{\frac{2\epsilon+1}{2}}$$

and consequently

$$\left(\frac{1}{n!c(\epsilon,R)^n}\right)^{\frac{2}{2\epsilon+1}} cap^{\frac{2}{2\epsilon+1}}(K,D) \geq \mathcal{L}^{4n}(K).$$

This gives the claim since when ϵ vary in $(0,\frac{1}{2})$ the exponent $\frac{2}{2\epsilon+1}$ vary in $(1,2)$.

Having the claim shown for compact sets the inequality follows easily, by taking the supremum, for any Borel set with the same constant. □

5.2 Local C^0 estimate and stability of solutions

In the previous chapter we have proven that any qpsh function belongs to L^p for $p < 2$ locally and that this is the optimal exponent. The lemma below gives the estimates on capacity and volume of sub-level sets for certain qpsh functions. In particular it shows that in the case of $u \in \mathcal{QPSH}(D)$ bounded near the boundary of D the local integrability of $|u|^p$ is ensured for $p < 2n$. Again this bound is optimal as the example of $-\frac{1}{\|q\|^2}$ shows. Consequently for such a function we can exclude the phenomenon about integrability of qpsh functions found in the previous chapter.

Lemma 5.2.1. *Let $u \in \mathcal{QPSH}(D) \cap L^\infty_{loc}(D)$ be such that*

$$\liminf_{q \to q_0} (u(q) - v(q)) \geq 0$$

for any $q_0 \in \partial D$ and some fixed $v \in \mathcal{QPSH}(D) \cap C(\overline{D})$. Then for any $p \in (1, 2)$ there exists a constant $C(p, diam(D))$, depending only on p and the diameter of D, such that for the sets

$$U(s) = \{u < v - s\} \subset D$$

we have

$$cap(U(s), D) \leq \frac{\int_D (\partial\partial_J u)^n}{s^n},$$

$$\mathcal{L}^{4n}(U(s)) \leq C(p, diam(D)) \frac{\int_D (\partial\partial_J u)^n}{s^{pn}}.$$

Proof. Take $\epsilon > 0$ and a compact $K \subset U(s)$. By definition one can find

$$w \in \mathcal{QPSH}(D) \cap L^\infty_{loc}(D)$$

such that $-1 \leq w \leq 0$ and

$$\int_K (\partial\partial_J w)^n \geq cap(K, D) - \epsilon.$$

Due to the way we have chosen K and the comparison principle, cf. Theorem 4.3.8,

$$cap(K, D) - \epsilon \leq \int_K (\partial\partial_J w)^n \leq \int_{\{\frac{u}{s} < \frac{v}{s} - 1\}} (\partial\partial_J w)^n \leq \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J w)^n$$

$$\leq \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} \left(\partial\partial_J(\frac{v}{s} + w)\right)^n \leq \frac{1}{s^n} \int_{\{\frac{u}{s} < \frac{v}{s} + w\}} (\partial\partial_J u)^n \leq \frac{\int_D (\partial\partial_J u)^n}{s^n}.$$

Letting ϵ tend to 0 and taking the supremum over all compacts K we obtain the first claim. The second one follows from Lemma 5.1.3. □

The next goal is to prove the a priori L^∞ estimate for the continuous solutions of the Dirichlet problem (II.5.1). This is the main result of this section whose proof is different from the one in the complex case. Firstly, note that by the result on the Dirichlet problem (II.5.1) with a continuous density and continuous boundary values, cf. [A03b, HL09c, HL20], we can find v solving

$$\begin{cases} (\partial\partial_J v)^n = 0, \\ v_{|\partial D} = \phi, \\ v \in C(\overline{D}). \end{cases} \tag{II.5.2}$$

By the characterization of the maximality of qpsh functions, as in [WZ15], this is the maximal qpsh function matching our boundary condition. For such a fixed v we define

$$U(s) = \{u < v - s\} \subset D \tag{II.5.3}$$

for the rest of this chapter and introduce the function

$$b(s) = \Big(cap(U(s), D)\Big)^{\frac{1}{n}}. \tag{II.5.4}$$

Theorem 5.2.2. *There exists a constant*

$$C\left(q, \| f \|_{L^q(D)}, \| \phi \|_{L^\infty(\partial D)}, diam(D)\right),$$

depending on q, $\| f \|_{L^q(D)}$, $\| \phi \|_{L^\infty(\partial D)}$ *and* $diam(D)$, *such that any solution* u *of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J u)^n = f\Omega_n \\ u_{|\partial D} = \phi \end{cases}$$

for $\phi \in C(\partial D)$, $f \in L^q(D)$ *and* $q > 2$, *satisfies*

$$\| u \|_{L^\infty(D)} \leq C.$$

Proof. Take any $s \geq 0$, $t \in [0,1]$ and $w \in \mathcal{QPSH}(D)$ such that $0 \leq w \leq 1$. Then

$$t^n \int\limits_{U(s+t)} (\partial\partial_J w)^n = \int\limits_{U(s+t)} (\partial\partial_J(tw - t - s))^n = \int\limits_{\{u<v-s-t\}} (\partial\partial_J(tw - t - s))^n$$

$$\leq \int\limits_{\{u<v-s+tw-t\}} (\partial\partial_J(tw - t - s))^n \leq \int\limits_{\{u<v-s+tw-t\}} (\partial\partial_J(v + tw - t - s))^n$$

$$\leq \int\limits_{\{u<v-s+tw-t\}} (\partial\partial_J u)^n \leq \int\limits_{\{u<v-s\}} (\partial\partial_J u)^n = \int\limits_{U(s)} (\partial\partial_J u)^n$$

due to inclusions of appropriate sets, superadditivity and the comparison principle, cf. Theorem 4.3.8. To conclude

$$t^n (b(s+t))^n \leq \int\limits_{U(s)} (\partial\partial_J u)^n.$$

Estimating the right hand side gives

$$\int\limits_{U(s)} (\partial\partial_J u)^n = \int\limits_{U(s)} f\Omega_n \leq \| f \|_{L^q(D)} \left(\int\limits_{U(s)} 1 d\mathcal{L}^{4n} \right)^{\frac{1}{q'}}$$

$$\leq \| f \|_{L^q(D)} C(p, diam(D)) \left(cap(U(s), D)\right)^{\frac{p}{q'}}$$

$$= \| f \|_{L^q(D)} C(p, diam(D))(b(s))^{n(1+\alpha(q))}$$

where we used Hölder's inequality and Lemma 5.1.3, p depends only on q' which is the conjugate of q and we choose it so that $\frac{p}{q'} > 1$. This reassembles to

$$tb(s+t) \leq A\left(q, \| f \|_{L^q(D)}, diam(D)\right) (b(s))^{1+\alpha(q)}$$

for any $s \geq 0$ and $t \in [0,1]$.

We would like to apply the De Giorgi lemma, stated below, for the function b. Let us just note that the condition (a) from the lemma is satisfied since for $s_n \searrow s$ the sets $U(s_n) \nearrow U(s)$ and under such an assumption $cap(U(s_n), D) \to cap(U(s), D)$, cf. [WK17]. The condition (b) follows from the first assertion of Lemma 5.2.1 as well as the dependence of s_0 only on q, $\| f \|_{L^q(D)}$ and $diam(D)$. Indeed, it was proven in Lemma 5.2.1 that

$$b^{\alpha(q)}(s) = cap(U(s), D)^{\frac{\alpha(q)}{n}} \leq \frac{\left(\int_D (\partial\partial_J u)^n\right)^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}}$$

$$= \frac{\| f \|_{L^1(D)}^{\frac{\alpha(q)}{n}}}{s^{\alpha(q)}} \leq \frac{c\left(q, \| f \|_{L^q(D)}, diam(D)\right)}{s^{\alpha(q)}}$$

so surely

$$s_0 \leq (2Ac)^{\frac{1}{\alpha(q)}}$$

and this estimate depends only on q, $\| f \|_{L^q(D)}$ and $diam(D)$. By Lemma 5.2.3 there exists

$$S\left(q, \| f \|_{L^q(D)}, diam(D)\right)$$

such that $b(s) = 0$ for any

$$s > S\left(q, \| f \|_{L^q(D)}, diam(D)\right).$$

This together with Lemma 5.1.3 gives our claim since then

$$\| u \|_{L^\infty} \leq sup|\phi| + S\left(q, \| f \|_{L^q(D)}, diam(D)\right)$$

$$= C\left(q, \| f \|_{L^q(D)}, \| \phi \|_{L^\infty(\partial D)}, diam(D)\right).$$

□

The technical result used above is the lemma below. It was used implicitly back in [K96], since then it turned out to be very useful while performing pluripotential estimates. Its form below is a combination of Lemma 1.5 in [GKZ08] and Lemma 2.4 in [EGZ09]. In [PSS12] it was attributed to De Giorgi. It is not so easy to find a reference with a correct statement and proof all together.

Lemma 5.2.3 (De Giorgi)**.** *Let $f : [0,\infty) \to [0,\infty]$ satisfy the following conditions:*
(a) f is right–continuous and non-increasing;
(b) $\lim_{x\to\infty} f(x) = 0$;
(c) there exist positive constants α, B such that for any $s \geq 0$ and $t \in [0,1]$

$$(*) \quad tf(t+s) \leq Bf(s)^{1+\alpha}.$$

Then there exists s_∞, depending on α, B and s_0 such that $f(s_0) \leq \frac{1}{2B}$, satisfying $f(s) = 0$ for any $s \geq s_\infty$. In fact one can choose s_∞ to be equal to $s_0 + \frac{2Bf(s_0)^\alpha}{1-2^{-\alpha}}$.

If in addition $(*)$ holds for all $s, t \geq 0$ then one can take even $s_\infty = \frac{2Bf(0)^\alpha}{1-2^{-\alpha}}$.

Proof. Fix any s_0 such that $f(s_0)^\alpha \leq \frac{1}{2B}$. Define inductively s_j unless $f(s_{j-1}) = 0$ as

$$s_j = \sup\{s > s_{j-1} \mid f(s) > \tfrac{f(s_{j-1})}{2}\}.$$

For any $s > s_j$ we have $f(s) \leq \frac{f(s_{j-1})}{2}$ so from right continuity $f(s_j) \leq \frac{f(s_{j-1})}{2}$ and s_j is the smallest time for which this happens. From $(*)$ it follows that $s_j \leq 1 + s_{j-1}$. For any $s \in (s_{j-1}, s_j)$

$$(s - s_{j-1})f(s) \leq Bf(s_{j-1})^{1+\alpha} < 2Bf(s)f(s_{j-1})^\alpha$$

since $f(s) \neq 0$ we conclude that

$$s - s_{j-1} < 2Bf(s_{j-1})^\alpha$$

for $s \in (s_{j-1}, s_j)$, consequently

$$s_j - s_{j-1} \leq 2Bf(s_{j-1})^\alpha \leq \tfrac{2Bf(s_0)^\alpha}{2^{\alpha(j-1)}}.$$

This show that one can take

$$s_\infty = s_0 + \sum_{j=1}^{\infty}(s_j - s_{j-1}) \leq s_0 + \sum_{j=1}^{\infty} \tfrac{2Bf(s_0)^\alpha}{2^{\alpha(j-1)}} = s_0 + \tfrac{2Bf(s_0)^\alpha}{1-2^{-\alpha}}.$$

Since $s_\infty \geq s_j$ for all j and $f(s_j) \leq \frac{f(s_0)}{2^j}$ it follows that $f(s) = 0$ for all $s \geq s_\infty$.

For the proof of the second assertion firstly observe that we may assume $f(0) \neq \infty$ since otherwise there is nothing to prove. We define s_j's inductively as before but starting from $s_0 = 0$. This time we do not know that $s_j \leq 1 + s_{j-1}$ but we do not need that since $(*)$ holds for all t. The claim then follows from the same argument as above. □

The L^∞ estimate allows us to prove the stability of solutions to the Dirichlet problem (II.5.1) in terms of the densities and boundary values. This will be needed for the proof of solvability of the Dirichlet problem (II.5.1) but is, of course, a result interesting in its own right. As we were told by Sławomir Dinew the idea of proving stability in the way presented in Proposition 5.2.5 is due to N. C. Nguyen. We start with a simple lemma.

Lemma 5.2.4. *There exists a constant C(q,diam(D)), depending on q and diam(D), such that any solution u of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J u)^n = f\Omega_n \\ u_{|\partial D} = 0 \end{cases}$$

for $f \in L^q(D)$ and $q > 2$, satisfies

$$\| u \|_{L^\infty(D)} \leq C(q, diam(D)) \| f \|_{L^q(D)}^{\frac{1}{n}} .$$

Proof. Suppose that $\| f \|_{L^q(D)} \neq 0$, otherwise there is nothing to prove. The function

$$v := \frac{u}{\|f\|_{L^q(D)}^{\frac{1}{n}}}$$

solves the Dirichlet problem

$$\begin{cases} v \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J v)^n = \frac{f}{\|f\|_{L^q(D)}}\Omega_n \\ v_{|\partial D} = 0 \end{cases} .$$

By Theorem 5.2.2 there exists a constant

$$C(q, diam(D)) := C(q, 1, 0, diam(D))$$

such that

$$\| v \|_{L^\infty(D)} \leq C(q, diam(D)),$$

this gives the claim. $\square$

Proposition 5.2.5. *There exists a constant C(q,diam(D)) such that if u and v satisfy*

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J u)^n = f\Omega_n \\ u_{|\partial D} = \phi \in C(\partial D) \end{cases} \quad \text{and} \quad \begin{cases} v \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J v)^n = g\Omega_n \\ v_{|\partial D} = \psi \in C(\partial D) \end{cases},$$

for $f, g \in L^q(D)$ and $q > 2$, then

$$\| u - v \|_{L^\infty(D)} \leq \sup_{\partial D} |\phi - \psi| + C(q, diam(D)) \| f - g \|_{L^q(D)}^{\frac{1}{n}}.$$

Proof. Consider a function w being the solution of

$$\begin{cases} w \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J w)^n = (f-g)_+\Omega_n \\ w_{|\partial D} = 0 \end{cases}.$$

Note that on ∂D we have $w + v + \inf(\phi - \psi) \leq u$ while

$$(\partial\partial_J (w + v + \inf(\phi - \psi)))^n \geq (f-g)_+ + g \geq f = (\partial\partial_J u)^n.$$

From the comparison principle, cf. Theorem 4.3.8,

$$w + v + \inf(\phi - \psi) \leq u$$

in $\overline{D}$ which, by Lemma 5.2.4, results in

$$u - v \geq w + \inf(\phi - \psi)$$

$$\geq -C(q, diam(D)) \| (f-g)_+ \|_{L^q(D)}^{\frac{1}{n}} - \sup |\phi - \psi|$$

$$\geq -C(q, diam(D)) \| f - g \|_{L^q(D)}^{\frac{1}{n}} - \sup |\phi - \psi|.$$

The same reasoning gives

$$v - u \geq -C(q, diam(D)) \| f - g \|_{L^q(D)}^{\frac{1}{n}} - \sup |\phi - \psi|.$$

This reassembles to our claim. $\square$

Remark 5.2.6. *Note that proofs of both the L^∞ estimate and the stability relies in [K05] on Theorem 4.3 therein. As we have seen above, in this context, that theorem is essentially equivalent to Lemma 5.2.3. This was noted before in [PSS12, K17].*

We prove another version, Theorem 5.2.8, of the stability estimate for the quaternionic Monge-Ampère equation – Theorem 5.2.5 above. It will not be of any use for the proof of the solvability of the Dirichlet problem (II.5.1) but it will be of an essential use in the next chapter. This result was proven in [KS20]. To be more precise, above we have shown that the uniform norm of the difference of solutions of (II.5.1) is under control by the uniform norm of the difference of boundary data and the L^q norm of the difference of the Monge-Ampère densities when they belong to L^q for $q > 2$. Our goal now is to prove that the uniform norm of the difference of solutions of (II.5.1) is under control by the L^p norm of that difference for appropriate p, cf. Theorem 5.2.8 below. From now on, whenever $u \in \mathcal{QPSH}(D)$ is locally bounded then writing $\| (\partial\partial_J u)^n \|_p$ automatically implies that we assume that the Borel measure $(\partial\partial_J u)^n$ has a density with respect to the Lebesgue measure $\mathcal{L}^{4n}$ and this density is in $L^p(D)$.

We show the main technical fact needed for the proof of the announced Theorem 5.2.8. The reasoning we perform in the rest of the section is based on the one presented in [GKZ08].

Proposition 5.2.7. *Fix $c_0 > 0$ and $p > 2$. Let $u, v \in \mathcal{QPSH}(D) \cap L^\infty_{loc}(D)$ be such that*

$$\liminf_{q \to q_0} (u - v)(q) \geq 0 \text{ for any } q_0 \in \partial D,$$

$$\| (\partial\partial_J u)^n \|_{L^p(D)} \leq c_0.$$

For any $0 < \alpha < \frac{p-2}{np}$ there exists a constant $C(c_0, \alpha, diam(D))$, depending on c_0, α and the diameter of D, such that for any $\epsilon > 0$

$$\sup(v - u) \leq \epsilon + C(c_0, \alpha, diam(D)) \left(cap(\{u - v < -\epsilon\}, D)\right)^\alpha.$$

Proof. Define

$$U_\epsilon(s) = \{u - v < -\epsilon - s\},$$

$$b_\epsilon(s) = (cap(U_\epsilon(s), D))^{\frac{1}{n}}$$

for $s \geq 0$ and $\epsilon > 0$. Firstly note that for all $t, s \geq 0$, $\epsilon > 0$ and $w \in \mathcal{QPSH}(D)$ such that $0 \leq w \leq 1$ one obtains from inclusions of sets, superadditivity and the comparison principle, cf. Theorem 1.2 in [WZ15],

$$t^n \int\limits_{U_\epsilon(s+t)} (\partial\partial_J w)^n = \int\limits_{U_\epsilon(s+t)} (\partial\partial_J (tw - t - s - \epsilon))^n = \int\limits_{\{u<v-s-t-\epsilon\}} (\partial\partial_J (tw - t - s - \epsilon))^n$$

$$\leq \int\limits_{\{u<v-s+tw-t-\epsilon\}} (\partial\partial_J (tw - t - s - \epsilon))^n \leq \int\limits_{\{u<v-s+tw-t-\epsilon\}} (\partial\partial_J (v + tw - t - s - \epsilon))^n$$

$$\leq \int\limits_{\{u<v-s+tw-t-\epsilon\}} (\partial\partial_J u)^n \leq \int\limits_{\{u<v-s-\epsilon\}} (\partial\partial_J u)^n = \int\limits_{U_\epsilon(s)} (\partial\partial_J u)^n.$$

From the Hölder inequality we obtain

$$\int\limits_{U_\epsilon(s)} (\partial\partial_J u)^n \leq \| (\partial\partial_J u)^n \|_{L^p(D)} \left(\mathcal{L}^{4n}(U_\epsilon(s))\right)^{\frac{1}{p'}}$$

$$\leq C(q', diam(D)) c_0 \left(cap(U_\epsilon(s), D)\right)^{\frac{q'}{p'}} = C(c_0, \alpha, diam(D))(b_\epsilon(s))^{n(1+n\alpha)},$$

where $q' \in (1,2)$ depends only on p' which is the conjugate of p and we choose it so that $\frac{q'}{p'} = 1 + n\alpha$. Since $1 + n\alpha < \frac{2}{p'}$ this is always possible. Taking the supremum over all w and n'th root of both sides gives

$$tb_\epsilon(s+t) \le C(c_0, \alpha, diam(D))(b_\epsilon(s))^{1+n\alpha}$$

for any $s, t \ge 0$ and $\epsilon > 0$. One easily checks, as in the proof of Theorem 5.2.2, that the function b_ϵ satisfies all the assumptions of Lemma 5.2.3. This gives

$$cap(\{u - v < -\epsilon - \frac{2C(c_0, \alpha, diam(D))}{1 - 2^{-n\alpha}} b_\epsilon(0)^{n\alpha}\}, D) = 0.$$

From the comparison of volume and capacity, cf. Lemma 5.1.3, it follows that

$$v - u \le \epsilon + C(c_0, \alpha, diam(D)) b_\epsilon(0)^{n\alpha}$$

almost everywhere in D. Since u and v are subharmonic we obtain that this holds in D, i.e.

$$\sup_D (v - u) \le \epsilon + C(c_0, \alpha, diam(D))(cap(U_\epsilon(0), D))^\alpha.$$

□

Theorem 5.2.8. *Fix $c_0 > 0$ and $p > 2$. Let $u,\ v \in \mathcal{QPSH}(D) \cap L^\infty_{loc}(D)$ be such that*

$$\liminf_{q \to q_0} (u - v)(q) \ge 0 \text{ for any } q_0 \in \partial D,$$

$$\| (\partial\partial_J u)^n \|_{L^p(D)} \le c_0.$$

For any $r \ge 1$ and

$$0 < \gamma < \frac{r}{r + np' + \frac{p'np}{p-2}} := \gamma_r$$

there exists a constant $C(c_0, \gamma, diam(D))$, depending only on c_0, γ and the diameter of D, such that

$$\sup_D (v - u) \le C(c_0, \gamma, diam(D)) \| (v - u)_+ \|^\gamma_{L^r(D)}.$$

Proof. First of all we may assume that $\| (v - u)_+ \|_{L^r(D)} \ne 0$ because otherwise the inequality holds with any constant $C(c_0, \gamma, diam(D))$. Arguing as in the beginning of the proof of Proposition 5.2.7 we obtain for any $\epsilon > 0$

$$cap(\{u - v < -2\epsilon\}, D) \le \epsilon^{-n} \int_{\{u - v < -\epsilon\}} (\partial\partial_J u)^n.$$

Since on the set $\{u - v < -\epsilon\}$ the function

$$\left(\left(\frac{v - u}{\epsilon}\right)_+\right)^{\frac{r}{p'}}$$

is bigger than 1, due to Hölder's inequality we may further estimate

$$\epsilon^{-n} \int_{\{u - v < -\epsilon\}} (\partial\partial_J u)^n \le \epsilon^{-n - \frac{r}{p'}} \int_D (v - u)_+^{\frac{r}{p'}} (\partial\partial_J u)^n$$

$$\leq \epsilon^{-n-\frac{r}{p'}} \| (\partial\partial_J u)^n \|_{L^p(D)} \left(\left(\int\limits_D ((v-u)_+)^r \right)^{\frac{1}{r}} \right)^{\frac{r}{p'}}$$

$$\leq \epsilon^{-n-\frac{r}{p'}} \| (\partial\partial_J u)^n \|_{L^p(D)} \| (v-u)_+ \|_{L^r(D)}^{\frac{r}{p'}} \leq \epsilon^{-n-\frac{r}{p'}} c_0 \| (v-u)_+ \|_{L^r(D)}^{\frac{r}{p'}} .$$

Applying Proposition 5.2.7 we get

$$\sup_D (v-u)$$

$$\leq 2\epsilon + C(c_0, \alpha, diam(D)) \left(cap(\{u - v < -2\epsilon\}, D)\right)^{\alpha}$$

$$\leq 2\epsilon + C(c_0, \alpha, diam(D)) \epsilon^{-\alpha n - \frac{r\alpha}{p'}} c_0^{\alpha} \| (v-u)_+ \|_{L^r(D)}^{\frac{r\alpha}{p'}}$$

for any $0 < \alpha < \frac{p-2}{np}$. Putting

$$\epsilon = \| (v-u)_+ \|_{L^r(D)}^{\gamma}$$

gives

$$\sup_D (v-u)$$

$$\leq 2 \| (v-u)_+ \|_{L^r(D)}^{\gamma} + C(c_0, \alpha, diam(D)) c_0^{\alpha} \| (v-u)_+ \|_{L^r(D)}^{\gamma\alpha(-n-\frac{r}{p'}) + \frac{r\alpha}{p'}} .$$

Choosing α such that

$$\gamma = \frac{r}{r + np' + \frac{p'}{\alpha}},$$

which is always possible since when α varies in

$$\left(0, \frac{p-2}{np}\right)$$

the quantity

$$\frac{r}{r + np' + \frac{p'}{\alpha}}$$

varies in $(0, \gamma_r)$, results in

$$\sup_D (v-u) \leq C(c_0, \gamma, diam(D)) \| (v-u)_+ \|_{L^r(D)}^{\gamma}$$

because

$$\gamma\alpha(-n - \frac{r}{p'}) + \frac{r\alpha}{p'} = \gamma \left(-\alpha n - \frac{\alpha r}{p'} + \frac{r\alpha}{\gamma p'} \right)$$

$$= \gamma \left(-\alpha n - \frac{\alpha r}{p'} + \frac{\alpha}{p'} \left(r + np' + \frac{p'}{\alpha} \right) \right) = \gamma.$$

□

5.3 Solving the Dirichlet problem

Theorem 5.3.1. *[Sr18] The Dirichlet problem*

$$\begin{cases} (\partial\partial_J u)^n = f\Omega_n \\ u_{|\partial D} = \phi \\ u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \end{cases}$$

in a smoothly bounded quaternionic strictly pseudoconvex domain D for $f \in L^q(D)$, $q > 2$ and $\phi \in C(\partial D)$ has a unique solution.

Proof. The uniqueness follows from the comparison principle – Theorem 4.3.8. For the solvability we take a sequence of continuous non-negative functions f_i converging to f in $L^q(D)$. Solving Dirichlet problems for them with our boundary condition, which is possible due to [A03b, HL09c, HL20], gives a sequence of continuous solutions u_i. Since, by Proposition 5.2.5, these solutions constitute a Cauchy sequence, in the uniform norm, it follows that u_i converge uniformly to some u, necessarily being qpsh. This is the solution we were looking for because of the convergence of the Monge-Ampère masses under the uniform convergence of functions as one can easily show from the definition, by induction for the currents $(\partial\partial_J u_i)^k$ for $k = 1, ..., n$, or by adding appropriate constants one can take a subsequence converging monotonically to u and apply Theorem 3.1 in [WW17]. □

The example below shows that the exponent two is optimal in this result, in the sense that for densities from $L^p(D)$ with $p < 2$ solutions may not even be bounded.

Proposition 5.3.2. *Let*

$$f : \mathbb{H}^n \ni q \longmapsto \log\big(\parallel q \parallel \big) \in \mathbb{R}.$$

It belongs to $\mathcal{QPSH}(\mathbb{H}^n)$ and

$$(\partial\partial_J f)^n = \frac{n!}{2 \parallel q \parallel^{2n}} \Omega_n.$$

Proof. We compute for

$$f_\epsilon(q) = \frac{1}{2} \log\left(\parallel q \parallel^2 + \epsilon \right).$$

$$\partial\partial_J f_\epsilon = \partial \left(\sum_{i=0}^{2n-1} (-1)^{i+1} (\partial_{\overline{z_{i+(-1)^i}}} f_\epsilon) dz_i \right) = \frac{1}{2}\partial \left(\sum_{i=0}^{2n-1} (-1)^{i+1} \frac{z_{i+(-1)^i}}{\parallel q \parallel^2 + \epsilon} dz_i \right)$$

$$= \frac{1}{2} \left(\sum_{i=0,j=0}^{2n-1} (-1)^{i+1} \frac{\delta^j_{i+(-1)^i}(\parallel q \parallel^2 + \epsilon) - z_{i+(-1)^i}\overline{z_j}}{(\parallel q \parallel^2 + \epsilon)^2} dz_j \wedge dz_i \right)$$

$$= \frac{1}{2}\Bigg(\sum_{i>j}^{2n-1} \Bigg((-1)^{i+1} \frac{\delta^j_{i+(-1)^i}(\parallel q \parallel^2 + \epsilon) - z_{i+(-1)^i}\overline{z_j}}{(\parallel q \parallel^2 + \epsilon)^2}$$

$$ - (-1)^{j+1} \frac{\delta^i_{j+(-1)^j}(\parallel q \parallel^2 + \epsilon) - z_{j+(-1)^j}\overline{z_i}}{(\parallel q \parallel^2 + \epsilon)^2} \Bigg) dz_j \wedge dz_i \Bigg)$$

$$= \frac{1}{2} \left(\sum_{i>j}^{2n-1} \left(\frac{2\delta^j_{i+(-1)^i}(\parallel q \parallel^2 + \epsilon) + (-1)^i z_{i+(-1)^i}\overline{z_j} + (-1)^{j+1} z_{j+(-1)^j}\overline{z_i}}{(\parallel q \parallel^2 + \epsilon)^2} \right) dz_j \wedge dz_i \right).$$

Let us denote by

$$M_{ij} = (-1)^i z_{i+(-1)^i}\overline{z_j} + (-1)^{j+1} z_{j+(-1)^j}\overline{z_i},$$

as in [WW17], and let

$$\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1}$$

be the sign of the permutation

$$(j_1, i_1, \dots, j_n, i_n) \to (0, 1, \dots, 2n-1).$$

With this notation we see that

$$2^n(\| q \|^2 + \epsilon)^{2n}(\partial\partial_J f_\epsilon)^n$$

$$= \left(\sum_{\substack{j_1,i_1,\dots,j_n,i_n:\\ \{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\, l\in\{1,\dots,n\}}} \left(\delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} \prod_{l\in\{1,\dots,n\}} \left(2\delta^{j_l}_{i_l+(-1)^{i_l}}(\| q \|^2 + \epsilon) + M_{i_l j_l}\right)\right)\right)\Omega_n$$

$$= \left(\binom{n}{0} \sum_{\{k_1,\dots,k_n\}=\{0,\dots,n-1\}} \delta^{(2k_1)(2k_1+1),\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^n (\| q \|^2 + \epsilon)^n \right.$$

$$+ \binom{n}{1} \sum_{\substack{\{j_1,i_1,2k_2,2k_2+1,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1\\ k_l\in\{0,\dots,n-1\}}} \delta^{j_1 i_1,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^{n-1}(\| q \|^2 + \epsilon)^{(n-1)} M_{i_1 j_1}$$

$$+ \binom{n}{2} \sum_{\substack{\{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,i_2>j_2\\ k_l\in\{0,\dots,n-1\}}} \delta^{j_1 i_1,j_2,i_2,\dots,(2k_n)(2k_n+1)}_{0,\dots,2n-1} 2^{n-2}(\| q \|^2 + \epsilon)^{(n-2)} M_{i_1 j_1} M_{i_2 j_2}$$

$$+ \dots$$

$$\left. + \binom{n}{n} \sum_{\substack{\{j_1,i_1,\dots,j_n,i_n\}=\{0,\dots,2n-1\}\\ i_l>j_l,\, l\in\{1,\dots,n\}}} \delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} \prod_{l\in\{1,\dots,n\}} M_{i_l j_l} \right)\Omega_n.$$

Note that for fixed indices $j_3, i_3, \dots, j_n, i_n$ the expression

$$M'_{j_3,i_3,\dots,j_n,i_n} = \sum_{\substack{j_1,i_1,j_2,i_2:\\ \{j_1,i_1,j_2,i_2,\dots,2k_n,2k_n+1\}=\{0,\dots,2n-1\}\\ i_1>j_1,i_2>j_2}} \delta^{j_1 i_1,\dots,j_n i_n}_{0,\dots,2n-1} M_{i_1 j_1} M_{i_2 j_2}$$

vanishes, as was already noticed in [WW17] for the purpose of computing the fundamental solution to the quaternionic Monge-Ampère operator. To see this let

$$\{0, \dots, 2n-1\} \setminus \{j_3, i_3, \dots, j_n, i_n\} = \{k, l, m, n\}$$

for $k > l > m > n$. Then

$$\frac{1}{2} M'_{j_3,i_3,\dots,j_n,i_n}$$

$$= \delta^{lk,nm,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{kl} M_{mn} + \delta^{mk,nl,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{km} M_{ln} + \delta^{nk,ml,j_3 i_3,\dots,j_n i_n}_{0,\dots,2n-1} M_{kn} M_{lm}$$

$$= \delta^{lk,nm,j_3i_3,\ldots,j_ni_n}_{0,\ldots,2n-1}\left(M_{kl}M_{mn} - M_{km}M_{ln} + M_{kn}M_{lm}\right)$$

$$= \pm\Big(\big((-1)^k z_{k+(-1)^k}\overline{z_l} + (-1)^{l+1} z_{l+(-1)^l}\overline{z_k}\big)\big((-1)^m z_{m+(-1)^m}\overline{z_n} + (-1)^{n+1} z_{n+(-1)^n}\overline{z_m}\big)$$

$$-\big((-1)^k z_{k+(-1)^k}\overline{z_m} + (-1)^{m+1} z_{m+(-1)^m}\overline{z_k}\big)\big((-1)^l z_{l+(-1)^l}\overline{z_n} + (-1)^{n+1} z_{n+(-1)^n}\overline{z_l}\big)$$

$$+\big((-1)^k z_{k+(-1)^k}\overline{z_n} + (-1)^{n+1} z_{n+(-1)^n}\overline{z_k}\big)\big((-1)^l z_{l+(-1)^l}\overline{z_m} + (-1)^{m+1} z_{m+(-1)^m}\overline{z_l}\big)\Big)$$

$$= \pm\Big((-1)^{k+m+1} z_{k+(-1)^k}\overline{z_n} z_{m+(-1)^m}\overline{z_l} + (-1)^{k+m} z_{k+(-1)^k}\overline{z_l} z_{m+(-1)^m}\overline{z_n}$$

$$+(-1)^{l+m+1} z_{l+(-1)^l}\overline{z_k} z_{m+(-1)^m}\overline{z_n} + (-1)^{m+l} z_{m+(-1)^m}\overline{z_k} z_{l+(-1)^l}\overline{z_n}$$

$$+(-1)^{k+l+1} z_{k+(-1)^k}\overline{z_m} z_{l+(-1)^l}\overline{z_n} + (-1)^{k+l} z_{k+(-1)^k}\overline{z_n} z_{l+(-1)^l}\overline{z_m}$$

$$+(-1)^{k+n+1} z_{k+(-1)^k}\overline{z_l} z_{n+(-1)^n}\overline{z_m} + (-1)^{k+n} z_{k+(-1)^k}\overline{z_m} z_{n+(-1)^n}\overline{z_l}$$

$$+(-1)^{m+n+1} z_{m+(-1)^m}\overline{z_k} z_{n+(-1)^n}\overline{z_l} + (-1)^{n+m} z_{n+(-1)^n}\overline{z_k} z_{m+(-1)^m}\overline{z_l}$$

$$+(-1)^{n+l+1} z_{n+(-1)^n}\overline{z_k} z_{l+(-1)^l}\overline{z_m} + (-1)^{l+n} z_{l+(-1)^l}\overline{z_k} z_{n+(-1)^n}\overline{z_m}\Big) = 0.$$

Because of that only the first two summands of the expression for $(\partial\partial_J f_\epsilon)^n$ do not vanish. We are left with

$$(\partial\partial_J f_\epsilon)^n$$

$$= \frac{1}{2^n(\| q \|^2 + \epsilon)^{2n}}\left(n!2^n(\| q \|^2 + \epsilon)^n - n!2^{n-1}(\| q \|^2 + \epsilon)^{(n-1)} \| q \|^2\right)\Omega_n$$

$$= \frac{n!(\| q \|^2 + 2\epsilon)}{2(\| q \|^2 + \epsilon)^{n+1}}\Omega_n.$$

Finally since the measures $(\partial\partial_J f_\epsilon)^n$ converge weakly to $(\partial\partial_J f)^n$, cf. Theorem 3.1 in [WW17], it is enough to find the weak limit of

$$\frac{n!(\| q \|^2 + 2\epsilon)}{2(\| q \|^2 + \epsilon)^{n+1}}$$

which by

$$\frac{n!(\| q \|^2 + 2\epsilon)}{2(\| q \|^2 + \epsilon)^{n+1}} \leq \frac{n!}{2 \| q \|^{2n}}$$

and Lebesgue's Dominated Convergence Theorem equals

$$\frac{n!}{2 \| q \|^{2n}}.$$

□

Chapter 6

Regularity of solutions

In this chapter we proceed to proving regularity of solutions to the Dirichlet problem for the quaternionic Monge-Ampère equation under the conditions on the boundary data and the density. This is the contents of Theorem 6.5. For that goal we firstly consider a more general situation in Theorem 6.1 below and then check that under the assumptions of Theorem 6.5 one can apply Theorem 6.1. All the results are taken from [KS20]. For this chapter for a domain D and $\alpha \in (0,1)$ we use the notation

$$Lip_{\alpha}\big(\overline{D}\big) := C^{0,\alpha}\big(\overline{D}\big).$$

Theorem 6.1. *Let $D \subset \mathbb{H}^n$ be a quaternionic strictly pseudoconvex domain. Suppose $f \in L^p(D)$ for $p > 2$ is a non-negative function, $\phi \in C(\partial D)$ and u is the solution to the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial \partial_J u)^n = f\Omega_n \\ u_{|\partial D} = \phi \end{cases}$$

such that $\Delta u(D)$ is finite. If there exists $0 < \nu < 1$ and $b \in Lip_{\nu}(\overline{D})$ such that

$$\begin{cases} b \leq u \text{ in } D \\ b = \phi \text{ on } \partial D \end{cases}$$

then

$$u \in Lip_{\alpha}(\overline{D})$$

for any $0 \leq \alpha < min\{\nu, 2\gamma_1\}$.

Remark 6.2. *Let us just emphasize that in the whole chapter we denote by Δu, for a locally bounded u, the distributional Laplacian. Since qpsh functions are in particular subharmonic for them Δu is a positive distribution, thus a measure, so it makes sense to write $\Delta u(D)$.*

We introduce the notation needed for the proof of this theorem. This approach is similar to the one presented in [N14]. For a fixed number $\delta > 0$ and a subharmonic u we consider

$$D_{\delta} = \{q \in D \mid dist(q, D) > \delta\},$$

$$u_{\delta}(q) = \sup_{\|p\| \leq \delta} u(q+p), \text{ for } q \in D_{\delta},$$

$$\hat{u_\delta}(q) = \frac{1}{\mathcal{L}^{4n}(B(0,1))\delta^{4n}} \int\limits_{\|q-p\|\leq\delta} u(p)d\mathcal{L}^{4n}(p), \text{ for } q \in D_\delta.$$

The lemma below is a composition of Lemmas 4.2 and 4.3 in [GKZ08]. As was noted by Nguyen in [N14] proofs given originally in [GKZ08] for plurisubharmonic functions used only subharmonicity.

Lemma 6.3. *[GKZ08] Let u be a subharmonic function in D. For a fixed $0 < \alpha < 1$ the following are equivalent*

(i) there exists $\delta_0, A > 0$ such that for any $0 < \delta \leq \delta_0$
$u_\delta - u \leq A\delta^\alpha$ in D_δ
(ii) there exists $\delta_1, B > 0$ such that for any $0 < \delta < \delta_1$
$\hat{u_\delta} - u \leq A\delta^\alpha$ in D_δ.

Moreover, there exists a constant c, depending only on the dimension n, such that for all $\delta > 0$ sufficiently small

$$\int\limits_{D_\delta} (\hat{u_\delta}(q) - u(q))\, d\mathcal{L}^{4n}(q) \leq c\Delta u(D)\delta^2.$$

Lemma 6.4. *Let u and b be as in Theorem 6.1. There exists a constant c_n such that for all $\delta > 0$ and $q \in \partial D_\delta$*

$$u_\delta(q) \leq u(q) + c_n\delta^\nu.$$

Proof. Denote by h the harmonic extension of $b_{|\partial D}$ to D. By the Proposition 2.4 in [N14] we know that $h \in Lip_\nu(\overline{D})$. Fix $q \in \partial D_\delta$ and take $p, p_0 \in \mathbb{H}^n$ such that $\| p \| = \| p_0 \| = \delta$, $u_\delta(q) = u(p+q)$ and $q + p_0 \in \partial D$. Since $b \leq u \leq h$ in $\overline{D}$, with equalities on ∂D, we obtain the following string of inequalities

$$\begin{aligned} u_\delta(q) - u(q) = u(p+q) - u(q) \leq h(p+q) - u(q) \leq h(p+q) - b(q) \\ = h(p+q) - h(q) + h(q) - b(q) \\ \leq \| h \|_{Lip_\nu(\overline{D})} \delta^\nu + h(q) - h(q+p_0) + b(q+p_0) - b(q) \\ \leq \left(2 \| h \|_{Lip_\nu(\overline{D})} + \| b \|_{Lip_\nu(\overline{D})}\right) \delta^\nu. \end{aligned}$$

□

Proof of Theorem 6.1. Fix $0 < \gamma < \gamma_1$ and consider the function

$$\tilde{u}_\delta(q) = \begin{cases} \max\{\hat{u}_\delta(q), u(q) + c_n\delta^\nu\} \text{ for } q \in D_\delta \\ u(q) + c_n\delta^\nu \text{ for } q \in \overline{D} \setminus D_\delta \end{cases}$$

which by Lemma 6.4 is qpsh in D as $\hat{u}_\delta \leq u_\delta$, cf. Proposition 2.1(4) in [WK17], and continuous in $\overline{D}$. Applying Theorem 5.2.8 for $u + c_n\delta^\nu$, $\tilde{u}_\delta$ and $r = 1$ we obtain

$$\sup_D(\tilde{u}_\delta - u - c_n\delta^\nu) \leq C\left(\| f \|_p, \gamma, diam(D)\right) \| (\tilde{u}_\delta - u - c_n\delta^\nu)_+ \|^\gamma_{L^1(D)}.$$

From the construction of $\tilde{u}_\delta$ it follows that the last inequality is equivalent to

$$\sup_{D_\delta}(\hat{u}_\delta - u - c_n\delta^\nu) \leq C\left(\| f \|_p, \gamma, diam(D)\right) \| (\hat{u}_\delta - u - c_n\delta^\nu)_+ \|^\gamma_{L^1(D_\delta)}.$$

Estimating further, by Lemma 6.3 and the trivial estimate

$$(\hat{u}_\delta - u - c_n\delta^\nu)_+ \leq \hat{u}_\delta - u,$$

we obtain for sufficiently small $\delta > 0$

$$\sup_{D_\delta}(\hat{u}_\delta - u - c_n\delta^\nu) \leq c^\gamma C\left(\| f \|_p, \gamma, diam(D)\right)(\Delta u(D))^\gamma \delta^{2\gamma}.$$

This results in

$$\sup_{D_\delta}(\hat{u}_\delta - u) \leq C(\| f \|_p, \gamma, diam(D), \Delta u(D), c, c_n)\delta^{\min\{\nu, 2\gamma\}}$$

for all δ sufficiently small. Since the constant C is independent of δ we obtain, again due to Lemma 6.3, that $u \in Lip_{\min\{\nu,2\gamma\}}(\overline{D})$. Since γ was arbitrary in $(0, \gamma_1)$ this gives our claim. □

Theorem 6.5. *Let $D \subset \mathbb{H}^n$ be a quaternionic strictly pseudoconvex domain. Suppose $f \in L^p(D)$ for $p > 2$ is a non-negative function bounded in a neighborhood of ∂D and $\phi \in C^{1,1}(\partial D)$. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J u)^n = f\Omega_n \\ u_{|\partial D} = \phi \end{cases}$$

is solvable and the unique solution is in $Lip_\alpha(\overline{D})$ for any $0 \leq \alpha \leq 2\gamma_1$.

Proof. The continuous solution u exists and is unique as was shown in Theorem 5.3.1. We need to check the assumptions of Theorem 6.1. For that goal we construct a function b as in Theorem 6.1 having in addition the properties that it is subharmonic and the Laplacian of it has the finite total mass. This will of course imply that the Laplacian of u has the finite total mass since b and u, both subharmonic, will agree on ∂D and $b \leq u$ in D. Take h to be the solution to the Dirichlet problem

$$\begin{cases} h \in \mathcal{QPSH}(D) \cap C(\overline{D}) \\ (\partial\partial_J h)^n = 0 \\ h_{|\partial D} = \phi. \end{cases}$$

By the comparison principle, cf. Theorem 1.2 in [WZ15], it is above any $v \in \mathcal{QPSH}(D) \cap C(\overline{D})$ such that $v_{|\partial D} = \phi$.

We will show that h is Lipschitz in $\overline{D}$ and its Laplacian has finite total mass in D. Suppose U is a neighborhood of $\overline{D}$ such that ϕ is extendable to a function $\hat{\phi} \in C^{1,1}(\overline{U})$. That is always possible due to Lemmas 6.37 and 6.38 in [GT01]. Consider a defining function ρ of D in the neighborhood V of $\overline{U}$, i.e.

$$\begin{cases} \rho \in \mathcal{QPSH}(V) \cap C^2(V) \\ D = \{\rho < 0\} \\ \rho_{|\partial D} = 0 \\ d\rho \neq 0 \text{ on } \partial D \\ (\partial\partial_J \rho)^n \geq \Omega_n \text{ on } \overline{D} \end{cases}.$$

We take A big enough such that $A\rho+\hat{\phi}$ and $A\rho-\hat{\phi}$ are in $\mathcal{QPSH}(U)$. Note that $A\rho+\hat{\phi} \leq h$ in D from the definition of h. This shows that the Laplacian of h has a finite total mass in D since $A\rho + \hat{\phi}$ has this property and the function $\hat{h}$ defined by

$$\hat{h}(q) = \begin{cases} h(q) \text{ for } q \in D \\ A\rho(q) + \hat{\phi}(q) \text{ for } q \in \overline{U} \setminus D \end{cases}$$

belongs to $\mathcal{QPSH}(U) \cap C^0(\overline{U})$, cf. Proposition 2.1(4) in [WK17]. Take $\epsilon > 0$ such that

$$D^{(\epsilon)} = \{q \in \mathbb{H}^n \mid dist(q, \overline{D}) < \epsilon\} \subset\subset U.$$

For any $p \in \partial D$ and q such that $\| q \| < \epsilon$ we have

$$\begin{aligned} &\hat{h}(p+q) \\ &\leq \hat{\phi}(p) + \max\{\| A\rho \pm \hat{\phi} \|_{C^1(\overline{U})}\} \| q \| \\ &= \phi(p) + \max\{\| A\rho \pm \hat{\phi} \|_{C^1(\overline{U})}\} \| q \| \end{aligned}$$

due to the mean value theorem and because $\hat{h} = h \leq \hat{\phi} - A\rho$ in D, as the subharmonic function $h - \hat{\phi} + A\rho$ attains its maximum, equal zero, on the boundary of D. Thus for

$$C = \max\{\| A\rho \pm \hat{\phi} \|_{C^1(\overline{U})}\},$$

every $p \in \partial D$ and $\| q \| < \epsilon$, we have

$$\hat{h}(p+q) - C \| q \| \leq \phi(p).$$

From the properties of h it means that for any $\| q \| < \epsilon$

$$\hat{h}(r+q) - C \| q \| \leq h(r)$$

for all $r \in \overline{D}$. Thus for $r \in \overline{D}$ and $\| q \| \leq \epsilon$ such that $r + q \in \overline{D}$ we obtain

$$\hat{h}(r+q) - h(r) = h(r+q) - h(r) \leq C \| q \|,$$

$$\hat{h}(r+q-q) - h(r+q) = h(r) - h(r+q) \leq C \| -q \|,$$

what results in

$$\| h(r+q) - h(r) \| \leq C \| q \|$$

for $r \in \overline{D}$ and $\| q \| < \epsilon$ such that $r + q \in \overline{D}$. This shows that h is locally Lipschitz continuous in $\overline{D}$ and consequently Lipschitz continuous.

By the assumptions for some $M > 0$ we have $f \leq M$ away from a compact $K \subset D$. Let u be the continuous solution to the Dirichlet problem from the statement. Take B big enough for the function $B\rho + h$ to be below u in a neighborhood of K and such that $B^n > M$. Then by superadditivity

$$\left(\partial\partial_J(B\rho + h)\right)^n \geq (\partial\partial_J B\rho)^n \geq f\Omega_n$$

at least in $D \setminus K$. The comparison principle implies that

$$B\rho + h \leq u$$

in $D \setminus K$ since the inequality holds on the boundary of this set.

We define $b = B\rho + h$. It is Lipschitz continuous and its Laplacian has finite total mass in D since h has these properties. □

Part III

Global case

Chapter 7

Quaternionic Monge-Ampère equation on HKT manifolds

7.1 Special hyperhermitian metric

This section is a natural continuation of the discussion carried out in Section 3.2, we keep the notation from Chapter 3. We wish to recall the two established classes of compatible metrics for quaternionic geometries discussed there.

Definition 7.1.1. *A hyperhermitian metric g on (M, I, J, K) is called hyperKähler, HK for short, if any of the following, equivalent (for simplicity suppose the manifold is simply connected), conditions are satisfied*

- $d\omega_I = d\omega_J = d\omega_K = 0$
- $\nabla^{LC}\omega_I = \nabla^{LC}\omega_J = \nabla^{LC}\omega_K = 0$
- $Hol(g) \subset Sp(m)$ *and* I, J, K *are induced by this holonomy group*
- (M, I, g), (M, J, g), (M, K, g) *are Kähler.*

Definition 7.1.2. *A Riemannian metric g compatible with the quaternionic structure (M, Q) is called quaternionic Kähler if $\nabla^{LC}Q \subset Q$.*

Remark 7.1.3. *The last definition for* $\dim M = 4$ *is too general as the quaternionic Kähler structures in this sense are just oriented Riemannian four manifolds. For this reason for* $\dim M = 4$ *one usually defines the notion of quaternionic Kähler structures separately.*

Those classes of metrics are standard to consider from the point of view of Berger's holonomy theorem on the irreducible holonomy groups of the metric connection. It follows from that theorem, with the later improvements, that $Sp(n)$ and $Sp(n) \cdot Sp(1)$, corresponding respectively to the hyperKähler and quaternionic Kähler metrics, are one of few infinite families which may occur, cf. [Be87, J00, GHJ03]. Those are the only occurring groups which correspond to the quaternionic geometries.

We abandon now the quaternionic non-hypercomplex side of the discussion. It is our objective to define the class of the so called HKT metrics being an intermediate class between general hyperhermitian and hyperKähler metrics. It is trivial to observe that for (M, I, J, K, g) being hyperKähler is equivalent to

$$d\Omega = 0.$$

Definition 7.1.4. *A hyperhermitian metric g on the hypercomplex manifold (M, I, J, K) is called HKT metric, where the abbreviation comes from hyperKähler with torsion, if*

$$\partial\Omega = 0. \tag{III.7.1}$$

Remark 7.1.5. *In this context, the form Ω is called an HKT form associated to an HKT metric g, via the isomorphism from Proposition 3.2.22. As we noted it is of the type $(2, 0)$ with respect to I. In [GP00] the definition of an HKT manifold is different. There, it is a hyperhermitian manifold for which a linear connection preserving g, I, J, K and having a skew-symmetric torsion tensor, of course after lowering the only upper index, exists. Due to their characteristic properties, from Section 3.1.1, this is equivalent to the equality of the three Bismut connections ∇^B_I, ∇^B_J and ∇^B_K for hermitian manifolds (M, I, g), (M, J, g) and (M, K, g) respectively. These conditions are equivalent to our definition as shown in Proposition 2 of [GP00]. It was proven there that the condition (III.7.1) encodes the equality of the torsions of the three mentioned Bismut connections and, as those are metric connections, they are uniquely determined by the torsion.*

These metrics emerged originally from mathematical physics. More exactly special connections occur naturally while studying the target space of certain sigma models in quantum theory, cf. [S86, HKLR87]. In the presence of the so called Wess-Zumino term and sufficient supersymmetry the HKT metrics appear, cf. [HP96]. It is worth noting that these metrics are interesting from the point of view of the differential geometry as well. An established mathematical treatment of basic properties of HKT manifolds is [GP00] which covers the early stage of research. It discusses in particular the relation of the HKT condition to the equality of the mentioned canonical connections of the induced hermitian structures and some elementary examples. The remark above demonstrates also that HKT manifolds are nothing but manifolds with a connection having holonomy contained in $Sp(n)$ and possessing a skew-symmetric torsion. Such connections appear naturally in string theory, cf. [S86]. From this description it follows also that HKT structures are natural differential geometric generalizations of HK structures where the torsion just vanishes. It is believed that HKT manifolds constitute the right analogue of Kähler manifolds in the hypercomplex world, cf. [V02, V09, GLV17]. This is partially due to the rigidity of HK manifolds, we know only two deformation classes in each dimension and two spontaneous examples due to O'Grady. One should note though that contrary to the name HKT manifolds are not in general Kähler at all. As the result of Verbitsky [V05] shows they can be Kähler only in the case when the manifold already admits HK metric. Let us also remark that originally it was conjectured, cf. [V02], that any hypercomplex manifold admits an HKT metric. This was disproved by Fino and Grantcharov in [FG04] and another examples, even simply connected ones by Swann [Sw10], followed.

In this string of arguments we present the result proven by Banos and Swann [BS04]. It generalizes the former partial results due to Poon and Swann [PS01] and due to Michelson and Strominger [MS00]. It is an analog of the existence of local potentials in Kähler case. In the form as below it was given in [AV06]. It will become handy when studying the quaternionic Monge-Ampère operator for HKT metrics.

Proposition 7.1.6. *Let (M, I, J, K) be a hypercomplex manifold and*

$$\Omega \in \Lambda^{2,0}_{I,\mathbb{R}}(M).$$

Locally, on M, the form Ω is given by

$$\Omega = \partial\partial_J f$$

for some smooth real function f if and only if

$$\partial\Omega = 0.$$

Let us elaborate on HKT geometry a bit more as it will be required to discuss the application of the quaternionic Monge-Ampère equation in hypercomplex geometry. In the next paragraph we will sometimes make assumptions on the triviality of the canonical bundle

$$K(M,I) := \Lambda_I^{2n,0}(M)$$

of HKT manifolds. We would like to discuss how restrictive this requirement is. First of all let us introduce the group $Sl_n(\mathbb{H})$. From one point of view we could define $Sl_n(\mathbb{H}) \subset Gl_n(\mathbb{H})$ as these matrices which have the Dieudonné determinant equal to 1. Since we have not introduced the Dieudonné determinant explicitly and this will be more natural for us, it may be equivalently defined as the group of those automorphisms of $\mathbb{H}^n$ preserving the canonical holomorphic volume form

$$dz_0 \wedge ... \wedge dz_{2n-1}.$$

Theorem 7.1.7. *[Ob56] Suppose (M, I, J, K) is a hypercomplex manifold. There exists a unique torsion free connection, denoted by ∇^{Ob}, on M such that*

$$\nabla^{Ob} I = \nabla^{Ob} J = \nabla^{Ob} K = 0.$$

Remark 7.1.8. *As we have mentioned in Section 3.2, in general it is not true that a hypercomplex manifold locally looks like $\mathbb{H}^n$, in the sense that there is no chart such that all the complex structures I, J, K look like in Example 3.2.13. Such structures were examined in [So75]. Existence of such trivializing charts is equivalent to the Obata connection being flat. The coordinate expression for this connection can be found in the mentioned paper by Obata himself [Ob56]. An invariant, global, formula can be found for example in Gauduchon's paper [G97b].*

The first part of the following theorem is obvious from the definition of the group $Sl_n(\mathbb{H})$ while the second one is due to Verbitsky.

Theorem 7.1.9. *[V07b] Suppose (M, I, J, K) is (for simplicity) a simply connected, hypercomplex manifold. Provided*

$$Hol(\nabla^{Ob}) \subset Sl_n(\mathbb{H}) \tag{III.7.2}$$

there exists a non-vanishing, I holomorphic, q-positive $(2n, 0)$ form Θ, i.e.

$$\Theta \in \Lambda^{2n,0}_{I,\mathbb{R}\ \geq 0}(M).$$

Partially conversely, if (M, I, J, K, g) is, in addition a compact HKT manifold, admitting I holomorphic non-vanishing $(2n, 0)$ form Θ then (III.7.2) holds.

We obtain from the above theorem that for a compact HKT manifold admitting non-vanishing I holomorphic q-positive $(2n, 0)$ form is equivalent to (III.7.2). Let us just note here that it is not known if more generally the triviality of the canonical bundle is equivalent to (III.7.2) for every compact hypercomplex manifold. This geometric situation is also important from the point of view of the generalization of the Berger holonomy theorem due to Merkulov and Schwachhöfer [MS99] for torsion free, not necessarily metric, connections. The group $Sl_n(\mathbb{H})$ is the one of a few additional families occurring in this context. Hypercomplex manifolds satisfying (III.7.2) are called an $Sl_n(\mathbb{H})$ manifolds.

7.2 Quaternionic Monge-Ampère equation in HKT geometry

We are ready to state the quaternionic version of the Calabi conjecture, as it was done in [AV10]. Firstly note that on a given HKT manifold, of quaternionic dimension n, the canonical bundle $K(M, I)$ is trivial topologically, e.g. Ω^n gives a smooth trivialization. The natural question presents itself whether every section can be obtained that way? The form Ω^n itself is, by far, not always holomorphic. Actually, unlike in the Kähler case where, up to the finite covering, topological triviality gives the holomorphic one, here the canonical bundle is not holomorphically trivial in many cases, e.g. Hopf surfaces. Arguably, cf. [V09], HKT manifolds with holomorphically trivial canonical bundle constitute a hypercomplex analogue of the Calabi–Yau manifolds. Alesker and Verbitsky proposed the following strategy to attack this trivialization problem, at least when the canonical bundle $K(M, I)$ is trivial holomorphically. Using the first order differential operator ∂_J, introduced by Verbitsky in [V02], they suggested to look for an HKT metric whose associated HKT form is

$$\Omega_\phi := \Omega + \partial\partial_J\phi$$

for some smooth real function ϕ and for which Ω_ϕ^n is the section we want to obtain. If such a ϕ exists then this new HKT metric g_ϕ can be obtained form Ω_ϕ by applying the isomorphism from Proposition 3.2.22. This approach allowed them to reformulate the described problem as a conjecture on the solvability of a certain PDE.

Conjecture 7.2.1. *[AV10] Let (M, I, J, K, g) be a compact HKT manifold. Suppose there exists a non-vanishing I holomorphic $(2n, 0)$ form Θ on M. For every smooth function f the quaternionic Monge-Ampère equation*

$$\begin{cases} (\Omega + \partial\partial_J\phi)^n = e^F\Omega^n \\ \Omega + \partial\partial_J\phi \geq 0 \end{cases} \tag{III.7.3}$$

has a unique, up to the constant, smooth solution provided the function f satisfies the necessary condition

$$\int_M (e^f - 1)\Omega^n \wedge \overline{\Theta} = 0.$$

The necessary normalizing condition on f follows from Stokes' theorem giving

$$\int_M \Omega^n \wedge \overline{\Theta} = \int_M \Omega_\phi^n \wedge \overline{\Theta}.$$

Note also that any non-vanishing, q-real $(2n, 0)$ form, if positive, is necessarily of the form

$$e^f\Omega^n$$

for some smooth f so in the conjecture above we truly take care of being able to obtain any section possible.

The question arises, like in the case of the complex Monge-Ampère equation on hermitian manifolds, why to look for a metric whose associated HKT form is a $\partial\partial_J\phi$ perturbation of the original one. This does not follow from a simple requirement of belonging to the de Rham class $[\Omega]_{dR}$ since in general the global $\partial\partial_J$ lemma is not true on an HKT manifold. It is true though for example for hyperKähler or, more generally, $Sl_n(\mathbb{H})$ manifolds, cf. [GLV17]. Being a $\partial\partial_J\phi$ perturbation of Ω becomes necessary if one agrees to look for solutions belonging to the class of Ω in a Bott-Chern type cohomology group

$$H^{2,0}_{BC}(M) := \frac{\{\eta \in \Lambda^{2,0}_I(M) \mid \partial\eta = \partial_J\eta = 0\}}{\partial\partial_J C^\infty(M)}$$

discussed in [GLV17]. This approach is additionally motived by the local result, Proposition 7.1.6, and the success of the similar naive approach in the case of some complex equations on non-Kähler manifolds. There the global $\partial\overline{\partial}$ lemma fails in general, yet one still looks for the $\partial\overline{\partial}\phi$ perturbations of the initial hermitian metric, cf. [TW10b, ChTW19].

Verbitsky argues in [V09] that HKT metrics giving the holomorphic trivialization constitute the analogue of Calabi–Yau metrics in this setting. These metrics have the property of being balanced with respect to any complex structure from S_M. They also fit in perfectly into the recently active stream of research on generalizations of Calabi-Yau spaces, the so called torsion Calabi–Yau manifolds, cf. [T15, Pi19].

Another conjecture, stated explicitly in [AS17], is obtained by dropping the assumption on the holomorphic triviality of the canonical bundle $K(M, I)$. It becomes natural to pose when comparing with the Calabi–Yau theorem [Y78], where you can solve the volume prescription problem regardless of the holomorphic triviality of the bundle.

Conjecture 7.2.2. *Let (M, I, J, K, g) be a compact HKT manifold. Given any q-positive, i.e. of the form $e^f\Omega^n$ for a smooth function f, trivialization Θ of $\Lambda^{2n,0}_I M$ is there an HKT metric $\tilde{g}$ such that the associated HKT form $\tilde{\Omega}$ satisfies*

$$\tilde{\Omega}^n = \Theta?$$

Equivalently does every complex volume form come from an HKT metric?

Of course from the discussion we had so far one may argue that this will be confirmed provided the following conjecture is true.

Conjecture 7.2.3. *Let (M, I, J, K, g) be a compact HKT manifold. For every smooth function f there exists exactly one real number b such that the quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\phi)^n = e^{(f+b)}\Omega^n$$

has a unique, up to the constant, smooth solution.

In the last conjecture the constant b has to appear, as for example in [TW10b], since there is no easy way to give an a priori necessary integrability condition for f.

It may be interesting to observe that equation (III.7.3) and more generally the one above encode questions of the type discussed in Section 3.1.1. It is not hard to note, from Example 3.2.13, that for a hyperhermitian manifold (M, I, J, K, g) and the constant c_n depending only on the dimension

$$\Omega^n \wedge \overline{\Omega}^n = c_n\omega_I^{2n}.$$

Consequently we see that if the quaternionic Monge-Ampère equation above is solvable for any f then

$$(\omega_{I,g_\phi})^{2n} = \frac{1}{c_n}(\Omega + \partial\partial_J\phi)^n \wedge (\overline{\Omega + \partial\partial_J\phi})^n = \frac{1}{c_n}e^{(2f+2b)}\Omega^n \wedge \overline{\Omega}^n = e^{F+C}\omega_I^{2n}$$

is satisfied with suitable ϕ and C for any F. Recalling the discussion curried out in Section 3.1.1 this means that any representative of $c_1^{BC}(M, I)$ can be obtained as the Chern-Ricci curvature of an HKT metric g_ϕ. This was noted already in [Ma11] for what Madsen calls the projected Chern form, cf. Section 7.1.3 in [Ma11].

In light of the success of generalizing the results from Kähler to even almost hermitian case in complex geometry, cf. [ChTW19], one could pose the following conjecture.

Conjecture 7.2.4. *Let (M, I, J, K, g) be a compact hyperhermitian manifold. For every smooth function f there exists exactly one real number b such that the quaternionic Monge-Ampère equation*

$$(\Omega + \partial\partial_J\phi)^n = e^{(f+b)}\Omega^n$$

has a unique, up to the constant, smooth solution.

To sum up we see that solving equation (III.7.3) under possibly most general assumptions is of crucial importance for answering quite naturally posed problems on HKT metrics.

7.3 Advances towards the proof

Let us now give an overview of the advances towards proving the conjectures stated in the proceeding section. The strategy is to use the so called continuity method. This reduces to exchanging equation (III.7.3) for a one parameter family of equations. This is obtained by perturbing the right hand side in a way which connects the desired right hand side with the one for which we know the solution exists, for example when the function f is constant. It is known that, in our case cf. [AV10, Ma11, A13], for this method to work the only issue is to obtain the so called a priori estimates. Those are estimates for the solution which depend only on the bound on the right hand sides of the equations in our family and the geometric quantities related to the hyperhermitian structure. This is so because with their aid we seek to show that the set of parameters for which we can solve the equations is closed. The openness of this set is much simpler in our setting and follows from the standard argument involving the implicit function theorem for suitably chosen Banach spaces and the operator between them. Consequently, choosing the family of equations in such a way that at least one of them is solvable shows that our original equation can be solved as well. Though this sounds elementary the struggle with obtaining the a priori estimates is severe. Due to the general theory of PDEs it is known that, by the Schauder estimates cf. [GT01], it is enough to obtain the $C^{2,\alpha}$ estimate for some $\alpha \in (0, 1)$. We now give the account of the results obtained towards that goal.

It is possible to obtain the C^0 estimate under the assumption of the existence of the holomorphic Θ, as in the section above, by repeating the Moser iteration method used by Yau in [Y78]. This was done by Alesker and Verbitsky in [AV10]. In [AS13] this bound was shown to hold when the hypercomplex structure is locally flat by using the local method of Błocki from [B11]. Under a stronger assumption that the HKT manifold admits a flat hyperKähler metric Conjecture 7.2.1 was confirmed by Alesker in [A13]. The assumption that the hyperKähler metric is flat, in the sense that the full Riemann curvature tensor vanishes, implies in particular that the hypercomplex structure is flat, as in this case the Obata connection coincides with the Levi-Civita one. Actually, the manifold is then a finite cover of a torus by Bieberbach's theorem on compact, flat Riemannian manifolds. Nevertheless under such an assumption Alesker proved the Laplacian bound and the analogue of the Evans-Krylov theorem, cf. [E82, T83]. This ensures the $C^{2,\alpha}$ bound since by this Evans-Krylov type theorem when the Laplacian of the solution is bounded then its $C^{2,\alpha}$ norm is automatically bounded as well. For the last theorem Alesker uses only the local flatness of the hypercomplex structure while for the Laplacian bound the existence of a flat HK metric was used.

Going to the general case, i.e. to the situation from Conjecture 7.2.3, only the C^0 bound is known. It was firstly obtained in [AS17]. Alesker and Shelukhin provided the proof of this estimate following the scheme of [B11]. It turned out though that the proof of one technical fact needed for the reasoning, Theorem 3.2.2 in [AS17], is surprisingly complicated and occupies a central part of that paper. What is more, their reasoning is based on the Douady space theory, Baston and Penrose transforms. All of these are completely different tools from the one usually applied in this kind of problems, cf. [Y78, TW10a, TW10b, ChTW19]. When it comes to the higher order estimates nothing is known. The difficulties come from the facts that the local situation on an HKT manifold does not correspond to the one in $\mathbb{H}^n$ due to a generic hypercomplex structure not being flat. More importantly, a big number of terms coming from differentiating quantities, usually used in arguments involving the maximum principle, with respect to the complex coordinates appears.

Motivated by this we give another, in our opinion simpler, proof of the C^0 estimate for this equation, in the next section. There are couple of features we wish to underline here. First of all, as we mentioned, our proof seems to be at least much shorter then the one from [AS17] which is nearly 50 pages long. Moreover we improve the estimate in the sense that the dependence of the bound from [AS17] on the right hand side is much weakened. We give an estimate which depends merely on the L^p norm of the right hand side for the suitable p. What is more, our methods are based on the geometric analytic techniques which seems to promise some success in obtaining also the higher order bounds in the general situation in the future.

Chapter 8

C^0 estimate

Theorem 8.1. *[Sr19] Let (M^n, I, J, K, g) be a compact HKT manifold and $F \in C^\infty(M)$. There exists a constant C depending only on the HKT structure, $q > 2n$ and*

$$\| e^F \|_{L^q},$$

such that for any smooth solution ϕ of the quaternionic Monge-Ampère equation

$$\begin{cases} (\Omega + \partial\partial_J \phi)^n = e^F \Omega^n \\ \Omega + \partial\partial_J \phi \geq 0 \\ \sup_M \phi = 0 \end{cases} \tag{III.8.1}$$

the following estimate holds

$$\| \phi \|_{L^\infty} \leq C.$$

We see that C depends in particular only on

$$\| e^F \|_{L^\infty}$$

and this in turn depends only on $\sup_M F$.

The proof of this theorem is strongly motivated by the reasoning performed in [TW10b] which is a refined version of the one described in [TW10a]. This in turn is based on an inequality obtained originally by Cherrier in [Ch87]. The method emerged in the course of proving the C^0 estimate for the complex Monge-Ampère equation on a compact hermitian, implicitly non-Kähler, manifold. The general strategy we take is as follows.

Firstly, in section 8.1, we prove the Cherrier type inequality – Lemma 8.1.1, cf. (22) in [Ch87] or Lemma 2.1 in [TW10b], for the assumed solution of (III.8.1). This is a cornerstone of the reasoning. Then, in Section 8.2, using the Moser iteration method we obtain a special bound on $\inf_M \phi$, Lemma 8.2.1, but still not the desired estimate since the right hand side depends on ϕ. From purely measure theoretic reasons this shows that the values of ϕ are separated from $\inf_M \phi$ by a positive constant, independent of ϕ as it turns out, on a set of a positive, independent of ϕ, measure, see Lemma 8.2.2. From this one can see the uniform bound follows easily provided we have at least an L^1 a priori estimate for which we refer to [AS13] where it was proven via the bounded Green function argument. We would like to point out that one can hope to try to adjust our argument also to the hyperhermitian not HKT case but this will certainly require some work. This is due to the fact that in this case in the formula (III.8.2) more bad terms will appear due to the fact that $\partial\alpha$ will not be zero necessarily and the whole argument will have to be adjusted accordingly.

In this chapter all the L^q norms are taken with respect to the volume element $\left(\Omega \wedge \overline{\Omega}\right)^n$. When we want to emphasize on what quantities the constant depends we put those in the brackets, e.g. $C(n, \| f \|_{L^1})$.

8.1 Cherrier type inequality

Lemma 8.1.1. *There exist positive constants C and p_0 both depending on the HKT geometry of the manifold, $q > 2n$ and*

$$\| e^F \|_{L^q}$$

such that for: any solution ϕ of (III.8.1), r being Hölder's conjugate of q and any $p \geq p_0$

$$\int_M |\nabla e^{-\frac{p}{2}\phi}|_g^2 \left(\Omega \wedge \overline{\Omega}\right)^n \leq Cp \| e^{-\phi} \|_{L^{pr}}^p .$$

Proof of Lemma 8.1.1. Let us define

$$\Omega_\phi = \Omega + \partial\partial_J\phi,$$

$$\alpha = \sum_{k=0}^{n-1} \Omega_\phi^k \wedge \Omega^{n-1-k},$$

$$\partial\left(\overline{\Omega^n}\right) = \beta \wedge \overline{\Omega^n} \text{ for some } (1,0) \text{ form } \beta.$$

Using the Stokes theorem we obtain that for any $p > 0$ and fixed $q > 2n$

$$\begin{aligned} C\left(\| e^F \|_{L^q}\right) \| e^{-\phi} \|_{L^{pr}}^p &\geq \int_M e^{-p\phi}(e^F - 1)\Omega^n \wedge \overline{\Omega^n} \\ &= \int_M e^{-p\phi}(\Omega_\phi^n - \Omega^n) \wedge \overline{\Omega^n} = \int_M e^{-p\phi}\partial\partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} \\ &= p\int_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} + \int_M e^{-p\phi}\partial_J\phi \wedge \beta \wedge \alpha \wedge \overline{\Omega^n}. \end{aligned} \tag{III.8.2}$$

First inequality above is the Hölder inequality and in the integration by parts we have used $\partial\alpha = 0$.

Our goal is to estimate the second factor on the right hand side of (III.8.2) which we reduce to finding a uniform pointwise bound on

$$\partial_J\phi \wedge \beta \wedge \alpha \wedge \overline{\Omega^n}. \tag{III.8.3}$$

This follows from the analogue of the inequality (2.2) in [TW10b], as in the lemma below.

Lemma 8.1.2. *There exists a positive constant B, depending on β, such that*

$$\left| \frac{\partial_J\phi \wedge \beta \wedge \Omega_\phi^k \wedge \Omega^{n-1-k}}{\Omega^n} \right| \leq \frac{B}{\epsilon} \frac{\partial\phi \wedge \partial_J\phi \wedge \Omega_\phi^k \wedge \Omega^{n-1-k}}{\Omega^n} + B\epsilon\frac{\Omega_\phi^k \wedge \Omega^{n-k}}{\Omega^n} \tag{III.8.4}$$

for any $\epsilon > 0$ and $k \in \{0, ..., n-1\}$.

Proof of Lemma 8.1.2. Let us note that, like in the complex case, one is able to simultaneously diagonalize, in a certain sense, both Ω and Ω_ϕ. Precisely we claim that for each $x \in M$ there exists a basis of $T^{1,0}_x M$, decomposition with respect to I, of the form

$$e_1, (\overline{e_1})J, ..., e_n, (\overline{e_n})J$$

such that

$$\Omega(e_i, e_j) = \Omega_\phi(e_i, e_j) = \Omega\left(e_i, (\overline{e_j})J\right) = \Omega_\phi\left(e_i, (\overline{e_j})J\right) = 0 \text{ for } i \neq j.$$

This follows from Lemma 8.1.3 below by taking $\Omega_1 = \Omega$ and $\Omega_2 = \Omega_\phi$.

Lemma 8.1.3. *Let Ω_1 be a strictly positive $(2,0)$ form, i.e.*

$$\Omega_1(z, \overline{z}J) > 0$$

for any non zero $(1,0)$ vector z, and Ω_2 a q-real $(2,0)$ form on M. For each $x \in M$ there exists a basis

$$e_1, (\overline{e_1})J, ..., e_n, (\overline{e_n})J$$

of $T^{1,0}_x M$ such that

$$\Omega_1(e_i, e_j) = \Omega_2(e_i, e_j) = \Omega_1\left(e_i, (\overline{e_j})J\right) = \Omega_2\left(e_i, (\overline{e_j})J\right) = 0 \textit{ for } i \neq j. \tag{III.8.5}$$

Proof of Lemma 8.1.3. Fix $x \in M$.

Take an orthonormal basis for Ω_1 like in Example 3.2.13, i.e. the basis

$$v_1, (\overline{v_1})J, ..., v_n, (\overline{v_n})J$$

such that (III.8.5) is satisfied for Ω_1 and in addition

$$\Omega_1(v_i, \overline{v_i}J) = 1 \text{ for } 1 \leq i \leq n.$$

With its aid one is able to check that the endomorphism

$$\widetilde{\Omega_2} : T^{1,0}_x M \longrightarrow T^{1,0}_x M$$

defined by the relation

$$\Omega_2(v, \cdot) = \Omega_1\left(\widetilde{\Omega_2}(v), \cdot\right)$$

is actually well defined since

$$\widetilde{\Omega_2}(v) = \sum_{1 \leq i \leq n} \left(\Omega_2(v, \overline{v_i}J)v_i - \Omega_2(v, v_i)\overline{v_i}J\right).$$

We prove by induction that for any $1 \leq k \leq n$ there exist linearly independent vectors

$$e_1, (\overline{e_1})J, ..., e_k, (\overline{e_k})J$$

and complex numbers

$$\lambda_1, ..., \lambda_k$$

such that (III.8.5) is satisfied and

$$\widetilde{\Omega_2}(e_i) = \lambda_i e_i \text{ for any } 1 \leq i \leq k.$$

For $k = 1$ take any eigenvector e_1 for $\widetilde{\Omega_2}$, it is linearly independent of $\overline{e_1}J$ and (III.8.5) is trivially satisfied.

Assume that the claim holds for a fixed $1 \leq k < n$ and take a set of vectors like in the statement for k. Let us note that for any $u, v \in T^{1,0}_x M$, using only the q-reality of Ω_1, Ω_2 and the definition of $\widetilde{\Omega_2}$, we obtain

$$\overline{\Omega_2(\overline{v}J, u)} = -\Omega_2\left(\overline{v}J, (\overline{\overline{u}J})\, J\right) = -(J\Omega_2)(\overline{v}, \overline{\overline{u}J}) = -\overline{\Omega_2}(\overline{v}, \overline{\overline{u}J}) = -\overline{\Omega_2(v, \overline{u}J)} = \\ -\overline{\Omega_1(\widetilde{\Omega_2}(v), \overline{u}J)} = -\overline{\Omega_1}\left(\overline{\widetilde{\Omega_2}(v)}, \overline{\overline{u}J}\right) = -(J\Omega_1)\left(\overline{\widetilde{\Omega_2}(v)}, \overline{\overline{u}J}\right) = -\Omega_1\left(\overline{\widetilde{\Omega_2}(v)}J, \overline{\overline{u}J}J\right) = \\ \Omega_1\left(\overline{\widetilde{\Omega_2}(v)}J, u\right)$$

thus proving that

$$\widetilde{\Omega_2}(\overline{v}J) = \overline{\widetilde{\Omega_2}(v)}J \text{ for any } v \in T^{1,0}_x M.$$

Since

$$\widetilde{\Omega_2}(e_i) = \lambda_i e_i,$$

by the above,

$$\widetilde{\Omega_2}(\overline{e_i}J) = \overline{\lambda_i}\,(\overline{e_i}J)$$

for $1 \leq i \leq k$. Consequently

$$\ker \Omega_1(e_i, \cdot) \subset \ker \Omega_2(e_i, \cdot) \text{ and } \ker \Omega_1(\overline{e_i}J, \cdot) \subset \ker \Omega_2(\overline{e_i}J, \cdot) \text{ for } 1 \leq i \leq k.$$

We introduce the following subspaces of $T^{1,0}_x M$

$$V = span\{e_1, (\overline{e_1})J, ..., e_k, (\overline{e_k})J\},$$
$$V' = \ker \Omega_1(e_1, \cdot) \cap \ker \Omega_1\left((\overline{e_1})J, \cdot\right) \cap ... \cap \ker \Omega_1(e_k, \cdot) \cap \ker \Omega_1\left((\overline{e_k})J, \cdot\right),$$
$$V'' = \ker \Omega_2(e_1, \cdot) \cap \ker \Omega_2\left((\overline{e_1})J, \cdot\right) \cap ... \cap \ker \Omega_2(e_k, \cdot) \cap \ker \Omega_2\left((\overline{e_k})J, \cdot\right).$$

Note that

$$T^{1,0}_x M = V \oplus V'$$

because

$$V \cap V' = \emptyset \text{ and } dim_{\mathbb{C}} V' \geq 2n - 2k.$$

Let us also observe that

$$\widetilde{\Omega_2}_{|V'} : V' \longrightarrow V'$$

since for $v \in V'$, by definition, $\widetilde{\Omega_2}(v)$ is such that

$$\Omega_2(v, \cdot) = \Omega_1\left(\widetilde{\Omega_2}(v), \cdot\right)$$

and $V' \subset V''$. Take e_{k+1} to be any eigenvector for $\widetilde{\Omega_2}_{|V'}$. Since $e_{k+1} \in V'$ and Ω_1 is q-real also $(\overline{e_{k+1}})J \in V'$. Finally due to the inclusion $V' \subset V''$ the linearly independent vectors

$$e_1, (\overline{e_1})J, ..., e_{k+1}, (\overline{e_{k+1}})J$$

satisfy (III.8.5) and thus all the required properties of the claim for $k + 1$. □

Remark 8.1.4. *In the setting under consideration this result simply states that given two hyperhermitian metrics one is able to trivialize one and diagonalize the other in the quaternionic basis. A similar statement, Proposition 3.2, is contained in [V10a] and justified by saying that it follows from "a standard argument which gives simultaneous digitalization of two pseudo-Hermitian forms". We note that at least one of Ω_1 or Ω_2 has to be positive.*

After normalization of e_i's we may assume that, at the point,

$$\Omega = e_1^* \wedge J^{-1}\left(\overline{e_1^*}\right) + ... + e_n^* \wedge J^{-1}\left(\overline{e_n^*}\right),$$

$$\Omega_\phi = \phi_1 e_1^* \wedge J^{-1}\left(\overline{e_1^*}\right) + ... + \phi_n e_n^* \wedge J^{-1}\left(\overline{e_n^*}\right) \text{ for some } \phi_i \geq 0.$$

Let us decompose

$$\beta = \sum_{i=1}^{n} b_{2i-1} e_i^* + b_{2i} J^{-1}(\overline{e_i^*}),$$

$$\partial\phi = \sum_{i=1}^{n} a_{2i-1} e_i^* + a_{2i} J^{-1}(\overline{e_i^*}),$$

then

$$\partial_J \phi = J^{-1}(\overline{\partial}\phi) = J^{-1}\left(\overline{\partial\phi}\right) = \sum_{i=1}^{n} -\overline{a_{2i}} e_i^* + \overline{a_{2i-1}} J^{-1}(\overline{e_i^*}).$$

Since b_i's are the coefficients of β in a unitary basis they are uniformly bounded by $|\beta|_g$. One easily checks the equalities

$$\Omega_\phi^k \wedge \Omega^{n-k} = \frac{k!(n-k)!}{n!} \sum_{1 \leq i_1 < ... < i_k \leq n} \phi_{i_1} ... \phi_{i_k} \Omega^n,$$

$$\partial\phi \wedge \partial_J \phi \wedge \Omega_\phi^k \wedge \Omega^{n-(k+1)}$$
$$= \frac{k!(n-k-1)!}{n!} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} |a_{2j-1}|^2 + |a_{2j}|^2 \right) \phi_{i_1} ... \phi_{i_k} \Omega^n,$$

$$\partial_J \phi \wedge \beta \wedge \Omega_\phi^k \wedge \Omega^{n-(k+1)}$$
$$= \frac{k!(n-k-1)!}{n!} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} -\overline{a_{2j}} b_{2j} - \overline{a_{2j-1}} b_{2j-1} \right) \phi_{i_1} ... \phi_{i_k} \Omega^n.$$

Thus we see that it is enough to prove that there exists B such that for any $0 \leq k < n$ and $\epsilon > 0$

$$\sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} |a_{2j}||b_{2j}| + |a_{2j-1}||b_{2j-1}| \right) \phi_{i_1} ... \phi_{i_k}$$
$$\leq B\epsilon(n-k) \sum_{1 \leq i_1 < ... < i_k \leq n} \phi_{i_1} ... \phi_{i_k} + \frac{B}{\epsilon} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} |a_{2j-1}|^2 + |a_{2j}|^2 \right) \phi_{i_1} ... \phi_{i_k}.$$

We have the string of inequalities following from the bound on b_i's and the AM–GM inequality

$$\sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} |a_{2j}||b_{2j}| + |a_{2j-1}||b_{2j-1}| \right) \phi_{i_1} ... \phi_{i_k}$$
$$\leq |\beta|_g \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} |a_{2j}| \phi_{i_1} ... \phi_{i_k} + |a_{2j-1}| \phi_{i_1} ... \phi_{i_k} \right)$$
$$\leq \frac{|\beta|_g}{2} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j \notin \{i_1, ..., i_k\}} 2\epsilon \phi_{i_1} ... \phi_{i_k} + \frac{|a_{2j}|^2 \phi_{i_1} ... \phi_{i_k} + |a_{2j-1}|^2 \phi_{i_1} ... \phi_{i_k}}{\epsilon} \right)$$

so we get that taking $B = |\beta|_g$ will do. $\square$

Having Lemma 8.1.2 established we are ready to deal with the term involving

$$\partial_J\phi \wedge \beta \wedge \alpha \wedge \overline{\Omega^n}$$

in the inequality (III.8.2).

Lemma 8.1.5. *There exist positive constants $C_1, ..., C_n, \epsilon_1, ..., \epsilon_n$ depending on the quantities listed in Lemma 8.1.1 such that*

$$\frac{p}{2^i}\int\limits_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} \leq C_i \parallel e^{-\phi} \parallel^p_{L^{pr}} + \epsilon C_i \sum_{k=1}^{n-i}\int\limits_M e^{-p\phi}\Omega^k_\phi \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \quad \text{(III.8.6)}$$

for all $i \in \{1, ..., n\}$, $\epsilon \in (0, \epsilon_i]$ and $p \geq p_i(\epsilon)$ a positive number depending on ϵ and i.

Proof of Lemma 8.1.5. We show the claim by induction for a fixed $q > 2n$.

For the case $i = 1$ let us note that from (III.8.4) there exists a uniform positive constant B such that for any $\epsilon > 0$ and $p > 0$

$$-\int\limits_M e^{-p\phi}\partial_J\phi \wedge \beta \wedge \alpha \wedge \overline{\Omega^n} = -\sum_{k=0}^{n-1}\int\limits_M e^{-p\phi}\partial_J\phi \wedge \beta \wedge \Omega^k_\phi \wedge \Omega^{n-1-k} \wedge \overline{\Omega^n}$$
$$\leq \sum_{k=0}^{n-1}\left(\frac{B}{\epsilon}\int\limits_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \Omega^k_\phi \wedge \Omega^{n-1-k} \wedge \overline{\Omega^n} + B\epsilon\int\limits_M e^{-p\phi}\Omega^k_\phi \wedge \Omega^{n-k} \wedge \overline{\Omega^n}\right).$$

We set $\epsilon_1 = 1$, then, by above, for any $\epsilon \leq \epsilon_1$ and $p \geq p_1(\epsilon) := \frac{2B}{\epsilon}$

$$-\int\limits_M e^{-p\phi}\partial_J\phi \wedge \beta \wedge \alpha \wedge \overline{\Omega^n}$$
$$\leq \frac{p}{2}\int\limits_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} + B\int\limits_M e^{-p\phi}\Omega^n \wedge \overline{\Omega^n} + \epsilon B\sum_{k=1}^{n-1}\int\limits_M e^{-p\phi}\Omega^k_\phi \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.$$

This in turn, coupled with the inequality (III.8.2) and Hölder's inequality, gives

$$\frac{p}{2}\int\limits_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} \leq C_1 \parallel e^{-\phi} \parallel^p_{L^{pr}} + \epsilon C_1 \sum_{k=1}^{n-1}\int\limits_M e^{-p\phi}\Omega^k_\phi \wedge \Omega^{n-k} \wedge \overline{\Omega^n}$$

proving the claim for $i = 1$.

For the inductive step suppose the claim holds for some fixed $1 \leq i < n$. To prove (III.8.6) for $i + 1$ we note that the LHS of (III.8.6) for i is twice the LHS of (III.8.6) for $i + 1$. Consequently it is enough to estimate the RHS of (III.8.6) for i by the LHS of (III.8.6) for $i + 1$ and the terms appearing on the RHS of (III.8.6) for $i + 1$. Note that since

$$\Omega_\phi = \Omega + \partial\partial_J\phi$$

we get

$$\epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\Omega_\phi^k \wedge \Omega^{n-k} \wedge \overline{\Omega^n}$$
$$= \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\Omega_\phi^{k-1} \wedge \Omega^{n-(k-1)} \wedge \overline{\Omega^n} \tag{III.8.7}$$
$$+ \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\partial\partial_J\phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n},$$

because of the form of the RHS of (III.8.6) for $i+1$ we only need to estimate the second summand. Applying Stokes' theorem and the fact that

$$\partial\overline{\Omega^n} = \beta \wedge \overline{\Omega^n}$$

gives

$$\epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\partial\partial_J\phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n}$$
$$= \epsilon p C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \tag{III.8.8}$$
$$+ \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\partial_J\phi \wedge \beta \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.$$

Below we bound both these summands. Let us set ϵ_{i+1} to be such that

$$\epsilon_{i+1} \leq \min\{\frac{1}{C_i 2^{i+2}}, \epsilon_i, 1\}$$

then for any $\epsilon \in (0, \epsilon_{i+1}]$ and $p \geq p_i(\epsilon)$

$$\epsilon p C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n}$$
$$\leq \frac{p}{2^{i+2}} \sum_{k=1}^{n} \int_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \tag{III.8.9}$$
$$= \frac{p}{2^{i+2}} \int_M e^{-p\phi}\partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n}.$$

For any $\epsilon \in (0, \epsilon_{i+1}]$ we set $p_{i+1}(\epsilon)$ to be such that

$$p_{i+1}(\epsilon) \geq \max\{p_i(\epsilon), 2^{i+2}BC_i\}$$

because then, again using firstly (III.8.4), for $p \geq p_{i+1}(\epsilon)$

$$\begin{aligned}
&\epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \partial_J \phi \wedge \beta \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \\
&\leq \epsilon C_i \sum_{k=1}^{n-i} \frac{B}{\epsilon} \int_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \\
&+ \epsilon C_i \sum_{k=1}^{n-i} B\epsilon \int_M e^{-p\phi} \Omega_\phi^{k-1} \wedge \Omega^{n-(k-1)} \wedge \overline{\Omega^n} \leq \frac{p}{2^{i+2}} \int_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \alpha \wedge \overline{\Omega^n} \\
&+ C_i B \int_M e^{-p\phi} \Omega^n \wedge \overline{\Omega^n} + \epsilon C_i B \sum_{k=1}^{n-(i+1)} \int_M e^{-p\phi} \Omega_\phi^k \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.
\end{aligned} \tag{III.8.10}$$

Note that for $\epsilon \in (0, \epsilon_{i+1}]$ and $p \geq p_{i+1}(\epsilon)$, from (III.8.6),

$$2\frac{p}{2^{i+1}} \int_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \alpha \wedge \overline{\Omega^n} \leq C_i \parallel e^{-\phi} \parallel_{L^{pr}}^p + \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \Omega_\phi^k \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.$$

By (III.8.7) the RHS of the above inequality equals to

$$C_i \parallel e^{-\phi} \parallel_{L^{pr}}^p + \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \Omega_\phi^{k-1} \wedge \Omega^{n-(k-1)} \wedge \overline{\Omega^n} + \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \partial\partial_J \phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.$$

Then by rewriting the second summand and applying (III.8.8) for the last one the above expression becomes

$$\begin{aligned}
&C_i \parallel e^{-\phi} \parallel_{L^{pr}}^p + \epsilon C_i \int_M e^{-p\phi} \Omega^n \wedge \overline{\Omega^n} + \epsilon C_i \sum_{k=1}^{n-(i+1)} \int_M e^{-p\phi} \Omega_\phi^k \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \\
&+ \epsilon p C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n} \\
&+ \epsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\phi} \partial_J \phi \wedge \beta \wedge \Omega_\phi^{k-1} \wedge \Omega^{n-k} \wedge \overline{\Omega^n}.
\end{aligned}$$

Applying (III.8.9) for the last but one summand, (III.8.10) for the last one and Hölder's inequality to bound

$$\parallel e^{-\phi} \parallel_{L^p}^p \text{ by } \parallel e^{-\phi} \parallel_{L^{pr}}^p$$

shows that this quantity is estimated by

$$C_{i+1} \parallel e^{-\phi} \parallel_{L^{pr}}^p + \epsilon C_{i+1} \sum_{k=1}^{n-(i+1)} \int_M e^{-p\phi} \Omega_\phi^k \wedge \Omega^{n-k} \wedge \overline{\Omega^n} + 2\frac{p}{2^{i+2}} \int_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \alpha \wedge \overline{\Omega^n},$$

for a constant C_{i+1} depending on B and C_i. We obtain (III.8.6) for $i+1$ and this finishes the proof of the inductive step. □

The proof of the main result, the Cherrier type inequality, is now finished by taking for a given $q > 2n$ in Lemma 8.1.5, $i = n$, $\epsilon = \epsilon_n$ and $p_0 = p_n(\epsilon_n)$ because then for any $p \geq p_0$

$$\begin{aligned}
&\int\limits_M |\nabla e^{-\frac{p}{2}\phi}|_g^2 \left(\Omega \wedge \overline{\Omega}\right)^n \\
&= n \int\limits_M \partial e^{-\frac{p}{2}\phi} \wedge \partial_J e^{-\frac{p}{2}\phi} \wedge \Omega^{n-1} \wedge \overline{\Omega^n} = \frac{np^2}{4} \int\limits_M e^{-p\phi} \partial\phi \wedge \partial_J \phi \wedge \Omega^{n-1} \wedge \overline{\Omega^n} \\
&\leq pC \left(\frac{p}{2^n} \int\limits_M e^{-p\phi} \partial\phi \wedge \partial_J\phi \wedge \alpha \wedge \overline{\Omega^n} \right) \leq pC \parallel e^{-\phi} \parallel_{L^{pr}}^p .
\end{aligned}$$

□

8.2 Proof of Theorem 8.1

Lemma 8.2.1. *There exist positive constants C and s_0, depending on the quantities listed in Lemma 8.1.1, such that for any solution of (III.8.1)*

$$e^{-s_0 \inf\limits_M \phi} \leq e^C \int\limits_M e^{-s_0\phi} \left(\Omega \wedge \overline{\Omega}\right)^n.$$

Proof of Lemma 8.2.1. From the Sobolev inequality for (M, g), cf. [A82], the fact that $\Omega^n \wedge \overline{\Omega^n}$ is uniformly comparable with the Riemannian volume element, Lemma 8.1.1 and the Hölder inequality we obtain

$$\left(\int\limits_M e^{-p\phi\gamma} \Omega^n \wedge \overline{\Omega^n} \right)^{\frac{1}{\gamma}} \leq C \left(\int\limits_M |\nabla e^{-\frac{p}{2}\phi}|_g^2 \Omega^n \wedge \overline{\Omega^n} + \int\limits_M e^{-p\phi} \Omega^n \wedge \overline{\Omega^n} \right) \leq pC \parallel e^{-\phi} \parallel_{L^{pr}}^p$$

for

$$r < \gamma := \frac{2n}{2n-1}$$

the Hölder conjugate of q, a uniform constant C and any $p \geq p_0$. This is equivalent to

$$\parallel e^{-\phi} \parallel_{L^{(p\frac{\gamma}{r})r}} = \parallel e^{-\phi} \parallel_{L^{p\gamma}} \leq (pC)^{\frac{1}{p}} \parallel e^{-\phi} \parallel_{L^{pr}} .$$

The iteration of the last inequality for

$$p_0, p_0\frac{\gamma}{r}, p_0\left(\frac{\gamma}{r}\right)^2,$$

and so on gives

$$\sup_M e^{-\phi} \leq C \parallel e^{-\phi} \parallel_{L^{p_0 r}}$$

hence we can take $s_0 := p_0 r$ since then

$$e^{-s_0 \inf\limits_M \phi} \leq C \int\limits_M e^{-s_0\phi} \Omega^n \wedge \overline{\Omega^n}.$$

□

Lemma 8.2.2. *[TW10a] There exist positive constants C_1 and C_2 such that for any solution of (III.8.1)*

$$\int_{\{\phi \leq \inf_M \phi + C_1\}} \left(\Omega \wedge \overline{\Omega}\right)^n \geq C_2.$$

Proof of Lemma 8.2.2. Having Lemma 8.2.1, the proof is exactly as in [TW10a]. The normalization of the volume element they use is purely for computational convenience. □

Proof of Theorem 8.1. As it has been said in order to finish the proof one needs at least an L^1 bound on ϕ. This estimate was shown in Proposition 2.3 in [AS13]. The proof is now finished by noting that either

$$\inf_M \phi + C_1 \geq 0 \text{ giving } -\inf_M \phi \leq C_1$$

or

$$C_3 \geq \| \phi \|_{L^1} \geq \int_{\{\phi \leq \inf_M \phi + C_1\}} |\phi| \left(\Omega \wedge \overline{\Omega}\right)^n \geq C_2 \left(-\inf_M \phi - C_1\right).$$

This gives a uniform constant

$$C = \max\{C_1, \frac{C_3}{C_2} + C_1\} = \frac{C_3}{C_2} + C_1$$

for which

$$-\inf_M \phi \leq C.$$

□

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